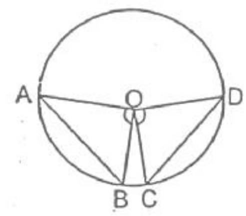
10.2 IMPORTANT THEOREMS

Theorem-1 : Equal chords of a circle subtend equal angles at the centre.



Given : AB and CD are the two equal chords of a circle with centre O.

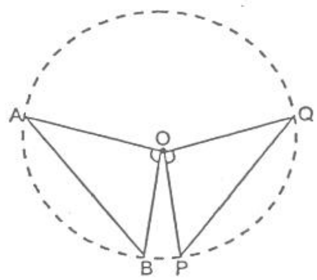
To Prove : $\angle AOB = \angle COD$.

Proof : In ΔAOB and ΔCOD,

$OA = OC$	[Radii of a circle]
$OB = OD$	[Radii of a circle]
$AB = CD$	[Given]
$\therefore\ \Delta AOB \cong \Delta COD$	[By SSS]
$\therefore\ \angle AOB = \angle COD$.	[By cpctc]

Converse of above Theorem :

In the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.



Given : $\angle AOB$ and $\angle POQ$ are two equal angles subtended by chords AB and PQ of a circle at its centre O.

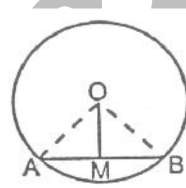
To Prove : $AB = PQ$

Proof : In ΔAOB and ΔPOQ,

$OA = OP$	[Radii of a circle]
$OB = OQ$	[Radii of a circle]
$\angle AOB = \angle POQ$	[Given]
$\therefore\ \Delta AOB \cong \Delta POQ$	[By SAS]
$\therefore\ AB = PQ$	[By cpctc]

Hence Proved.

Theorem-2 : The perpendicular from the centre of a circle to a chord bisects the chord.



Given : A circle with centre O. AB is a chord of this circle. $OM \perp AB$.
To Prove : $MA = MB$.
Construction : Join OA and OB.

Proof : In right triangles OMA and OMB,

	OA = OB	[Radii of a circle]	
	OM = OM	[Common]	
	$\angle$OMA = $\angle$OMB	[90^0 each]	
$\therefore$	ΔOMA $\cong$ ΔOMB	[By RHS]	
$\therefore$	MA = MB	[By cpctc]	**Hence Proved.**

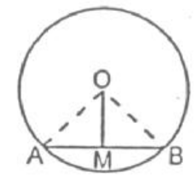
Converse of above Theorem :

The line drawn through the centre of a circle to bisect a chord a perpendicular to the chord.

Given : A circle with centre O. AB is a chord of this circle whose mid-point is M.

To Prove : OM $\perp$ AB.

Construction : Join OA and OB.

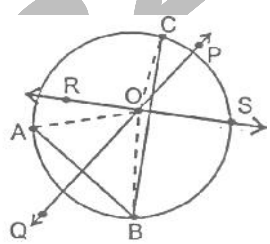


Proof : In ΔOMA and ΔOMB.

	MA = MB	[Given]
	OM = OM	[Common]
	OA = OB	[Radii of a circle]
$\therefore$	ΔOMA $\cong$ ΔOMB	[By SSS]
$\therefore$	$\angle$AMO = $\angle$BMO	[By cpctc]
But	$\angle$AMO + $\angle$BMO = 180^0	[Linear pair axiom]
$\therefore$	$\angle$AMO = $\angle$BMO = 90^0	
$\Rightarrow$	OM $\perp$ AB.	

Theorem-3 : There is one and only one circle passing through three given non-collinear points.

Proof : Let us take three points A, B and C, which are not on the same line, or in other words, they are not collinear [as in figure]. Draw perpendicular bisectors of AB and BC say, PQ and RS respectively. Let these perpendicular bistros intersect at one point O. (Note that PQ and RS will intersect because they are not parallel) [as in figure].



$\therefore$ O lies on the perpendicular bisector PQ of AB.

$\therefore$ OA = OB

[$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

Similarly,

$\therefore$ O lies on the perpendicular bisector RS of BC.

$\therefore$ OB = OC

[$\because$ Every point on the perpendicular bisector of a line segment is equidistant from its end points]

So, OA = OB = OC

i.e., the points A, B and C are at equal distances from the point O.

So, if we draw a circle with centre O and radius OA it will also pass through B and C. This shows that there is a circle passing through the three points A, B and C. We know that two lines (perpendicular bisectors) can intersect at only one point, so we can draw only one circle with radius OA. In other words, there is a unique circle passing through A, B and C. **Hence Proved.**

REMARK :

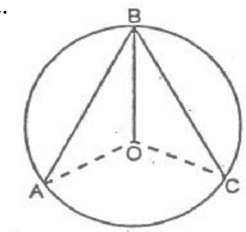
If ABC is a triangle, then by above theorem, there is a unique circle passing through the three vertices A, B and C of the triangle. This circle the circumcircle of the ΔABC. Its centre and radius are called respectively the circumcentre and the circumradius of the triangle.

Ex.1 In figure, AB = CB and O is the centre of the circle. Prove that BO bisects ∠ABC.

Sol. **Given :** In figure, AB = CB and O is the centre of the circle.

To Prove : BO bisects ∠ABC.

Construction : Join OA and OC.



Proof : In ΔOAB and ΔOCB,

OA = OC	[Radii of the same circle]
AB = CB	[Given]
OB = OB	[Common]
∴ ΔOAB ≅ ΔOCB	[By SSS]
∴ ∠ABO = ∠CBO	[By cpctc]

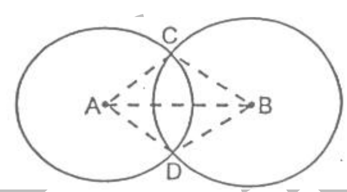
⇒ BO bisects ∠ABC. **Hence Proved.**

Ex.2 Two circles with centres A and B intersect at C and D. Prove that ∠ACB = ∠ADB.

Sol. **Given :** Two circles with centres A and B intersect at C and D.

To Prove : ∠ACB = ∠ADB.

Construction : Join AC, AD, BC, BD and AB.



Proof : In ΔACB an ΔADB,

AC = AD	[Radii of the same circle]
BC = BD	[Radii of the same circle]
AB = AB	[Common]
∴ ΔACB ≅ ΔADB	[By SSS]
∴ ∠ACB = ∠ADB.	[By cpctc]

Hence Proved.

E.3 In figure, AB ≅ ACand O is the centre of the circle. Prove that OA is the perpendicular bisector of BC.

Sol. **Given :** In figure, AB ≅ AC and O is the centre of the circle.

To Prove : OA is the perpendicular bisector of BC.

Construction : Join OB and OC.

Proof :

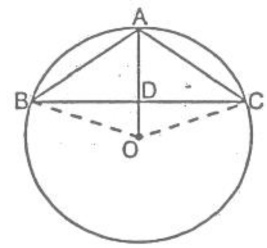
∴ AB ≅ AC **[Given]**

∴ **chord AB = chord AC.**

[∵ If two arcs of a circle are congruent, then their corresponding chords are equal.]

∴ ∠AOB = ∠AOC ....(i) [∵ Equal chords of a circle subtend equal angles at the centre]

In ΔOBC and ΔOCD,



$\angle DOB = \angle DOC$ [From (1)]

$OB = OC$ [Radii of the same circle]

$OD = OD$ [Common]

$\therefore$ $\Delta OBD \cong \Delta OCD$ [By SAS]

$\therefore$ $\angle ODB = \angle ODC$ ....(ii) [By cpctc]

And $BD = CD$ ...(ii) [By cpctc]

But $\angle BDC = 180^0$

$\therefore$ $\angle ODB + \angle ODC = 180^0$

$\Rightarrow$ $\angle ODB + \angle ODB = 180^0$ [From equation (ii)]

$\Rightarrow$ $2\angle ODB = 180^0$

$\Rightarrow$ $\angle ODB = 90^0$

$\therefore$ $\angle ODB = \angle ODC = 90^0$ ....(iv) [From (ii)]

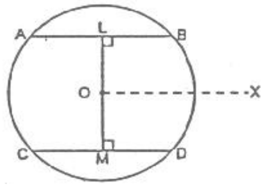
So, by (iii) and (iv), OA is the perpendicular bisector of BC. **Hence Proved.**

Ex.4 Prove that the line joining the mid-points of the two parallel chords of a circle passes through the centre of the circle.

Sol. Let AB and CD be two parallel chords of a circle whose centre is O.

Let l and M be the mid-points of the chords AB and CD respectively. Join PL and OM.

Draw $OX \parallel AB$ or CD.



$\therefore$ L is the mid-point of the chord AB and O is the centre of the circle

$\therefore$ $\angle OLB = 90^0$

[$\because$ The perpendicular drawn from the centre of a circle to chord bisects the chord]

But, $OX \parallel AB$

$\therefore$ $\angle LOX = 90^0$......(i)

[$\because$ Sum of the consecutive interior angles on the same side of a transversal is 180^0]

$\therefore$ M is the mid-point of the chord CD and O is the centre of the circle.

$\therefore$ $\angle OMD = 90^0$

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

But $OX \parallel CD$ ....(ii)

[$\because$ Sum of the consecutive interior angles on the same side of a transversal is 180^0]

$\therefore$ $\angle MOX = 90^0$

From above equations, we get

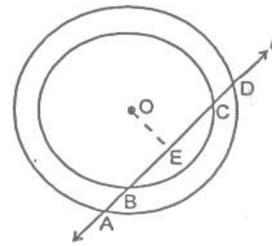
$\angle LOX + \angle MOX = 90^0 + 90^0 = 180^0$

$\Rightarrow$ $\angle LOM = 180^0$

$\Rightarrow$ LM is a straight line passing through the centre of the circle. **Hence Proved.**

Ex.5 ℓ is a line which intersects two concentric circle (i.e., circles with the same centre) with common centre O at A, B, C and D (as in figure). Prove that AB = CD.

Sol. **Given :** ℓ is a line which intersects two concentric circles (i.e., circles with the same centre) with common centre O at A, B, C and D.



To Prove : AB = CD.

Construction : Draw OE $\perp \ell$

Proof :

$\therefore$ The perpendicular drawn from the centre of a circle to a chord bisects the chord

$\therefore$ AE = ED .....(i)

And BE = EC ....(ii)

Subtracting (ii) from (i), we get

AE - BE = ED - EC

$\Rightarrow$ AB = CD. **Hence Proved.**

Ex.6 PQ and RS are two parallel chords of a circle whose centre is O and radius is 10 cm. If PQ = 16 cm and RS = 12 cm, find the distance between PQ and RS, if they lie.

(i) on the same side of the centre O.

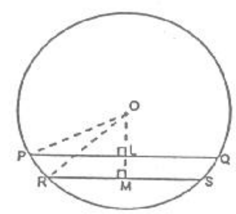
(ii) on opposite sides of the centre O.

Sol. (i) Draw the perpendicular bisectors OL and OM of PQ and RS respectively.

$\therefore$ PQ $\parallel$ RS

$\therefore$ OL and OM are in the same line.

$\Rightarrow$ O, L and M are collinear.



Join OP and OR.

In right triangle OLP,

$OP^2 = OL^2 + PL^2$ [By Pythagoras Theorem]

$\Rightarrow (10)^2 = OL^2 + \left(\frac{1}{2}\times pq\right)^2$

[$\therefore$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$\Rightarrow 100 = OL^2 + \left(\frac{1}{2}\times 16\right)^2$

$\Rightarrow 100 = OL^2 + (8)^2$

$\Rightarrow 100 = OL^2 + 64$

$\Rightarrow OL^2 = 100 - 64$

$\Rightarrow OL^2 = 36 = (6)^2$

$\Rightarrow$ OL = 6 cm

In right triangle OMR,

$OR^2 = OM^2 + RM^2$ **[By Pythagoras Theorem]**

$\Rightarrow \quad OR^2 = OM^2 + \left(\frac{1}{2} \times RS\right)^2$

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$\Rightarrow \quad (10)^2 = OM^2 + \left(\frac{1}{2} \times 12\right)^2$

$\Rightarrow \quad (10)^2 = OM^2 + (6)^2$

$\Rightarrow \quad OM^2 = (10)^2 - (6)^2 = (10 - 6)(10 + 6) = (4)(16) = 64 = (8)^2$

$\Rightarrow \quad OM = 8$ cm

$\therefore \quad LM = OM - OL = 8 - 6 = 2$ cm

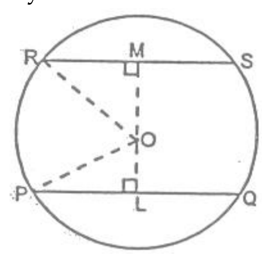
Hence, the distance between PQ and RS, if they lie on he same side of the centre O, is 2 cm.

(ii) Draw the perpendicular bisectors OL and OM of PQ and RS respectively.

$\therefore \quad PQ \parallel RS$

$\therefore \quad$ OL and OM are in the same line

$\Rightarrow \quad$ L, O and M are collinear.



Join OP nd OR.

In right triangle OLP,

$OP^2 = OL^2 + PL^2$ [By Pythagoras Theorem]

$\Rightarrow \quad OP^2 = OL^2 + \left(\frac{1}{2} \times pQ\right)^2$

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$\Rightarrow \quad (10))^2 = OL^2 + \left(\frac{1}{2} \times 16\right)^2$

$\Rightarrow \quad 100 = OL^2 + (8)^2$

$\Rightarrow \quad 100 = OL^2 + 64$

$\Rightarrow \quad OL^2 = 100 - 64$

$\Rightarrow \quad OL^2 = 36 = (6)^2$

$\Rightarrow \quad OL = 6$ cm

In right triangle OMR,

$OR^2 = OM^2 + RM^2$ [By Pythagoras Theorem]

$\Rightarrow \quad OR^2 = OM^2 + \left(\frac{1}{2} \times 12\right)^2$

[$\because$ The perpendicular drawn from the centre of a circle to a chord bisects the chord]

$\Rightarrow \quad (10)^2 = OM^2 + \left(\frac{1}{2} \times RS\right)^2$

$\Rightarrow \quad (10)^2 = OM^2 + (6)^2$

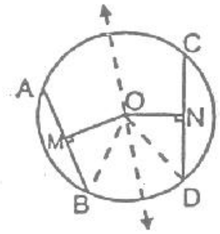
$\Rightarrow \quad OM^2 = (10)^2 - (6)^2 = (10 - 6)(10 + 6) = (4)(16) = 64 = (8)^2$

$\Rightarrow \quad OM = 8$ cm

$\therefore \quad LM = OL + OM = 6 + 8 = 14$ cm

Hence, the distance between PQ and RS, if they lie on the opposite side of the centre O, is 14 cm.

Theorem-4 : Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).



Given : A circle have two equal chords AB & CD. .e. AB = CD and OM $\perp$ AB, ON $\perp$ CD

To Prove : OM = ON

Construction : Join OB & OD

Proof : AB = CD (Given)

[$\because$ The perpendicular drawn from the centre of a circle to bisect the chord.]

$\therefore \quad \frac{1}{2}AB = \frac{1}{2}CD$

$\Rightarrow$ BM = DN

In ΔOMB & ΔOND

$\angle OMB = \angle OND = 90^0$ [Given]

OB = OD [Radii of same circle]

Side BM = Side DN [Proved above]

$\therefore$ $\Delta OMB \cong \Delta OND$ [By R.H.S.]

$\therefore$ OM = ON [By cpctc] **Hence Proved.**

REMARK :

Chords equidistant from the centre of a circle are equal in length.

Ex.7 AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E. Prove that EB = ED.

Sol. **Given :** AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E.

To Prove : EB = ED.

Construction : From O draw OP $\perp$ AB and OQ $\perp$ CD. Join OE.

Proof : $\therefore$ AB = CD [Given]

$\therefore$ OP = OQ [$\therefore$ Equal chords of a circle are equidistant from the centre]

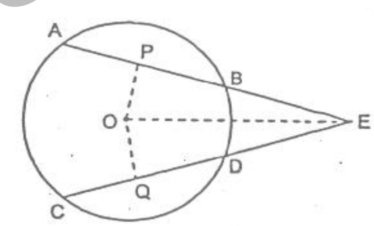
Now in right tingles OPE and OQE,

OE = OE [Common]

Side OP = Side OQ [Proved above]

$\therefore$ $\Delta OPE \cong \Delta OQE$ [By RHS]

$\therefore$ OE = QE [By cpctc]



$\Rightarrow$ PE - $\frac{1}{2}$AB = QE - $\frac{1}{2}$CD [$\therefore$ AB = CD (Given)]

$\Rightarrow$ PE - PB = QE - QD

$\Rightarrow$ EB = ED. **Hence Proved.**

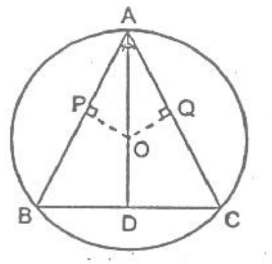
Ex.8 Bisector AD of ∠BAC of ΔABC passed through the centre O of the circumcircle of ΔABC. Prove that AB = AC.

Sol. **Given :** Bisector AD of ∠BAC of ΔABC passed through the centre O of the circumcircle of ΔABC,

To Prove : AB = AC.

Construction : Draw OP ⊥ AB and OQ ⊥ AC.

Proof :



In ΔAPO and ΔAQO,

∠OPA = ∠OQA [Each = 90^0 (by construction)]

∠OAP = ∠OAQ [Given]

OA = OA [Common]

∴ ΔAPO ≅ ΔAQO [By ASS cong. prog.]

∴ OP = OQ [By cpctc]

∴ AB = AC. [∵ Chords equidistant from the centre are equal] **Hence Proved.**

Ex.9 AB and CD are the chords of a circle whose centre is O. They intersect each other at P. If PO be the bisector of ∠APD, prove that AB = CD.

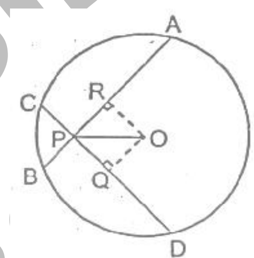
OR

In the given figure, O is the centre of the circle and PO bisect the angle APD. prove that AB = CD.

Sol. **Given :** AB and CD are the chords of a circle whose centre is O. They interest each other at P. PO is the bisector of ∠APD.

To Prove : AB = CD.

Construction : Draw OR ⊥ AB and OQ ⊥ CD.



Proof : In ΔOPR and ΔOPQ,

∠OPR = ∠OPQ [Given]

OP = OP [Common]

And ∠ORP = ∠OQP [Each = 90^0]

∴ ΔORP ≅ ΔOPQ [By AAS]

∴ OR = OQ [By cpctc]

∴ AB = CD [∵ Chords of a circle which are equidistant from the centre are equal]

REMARK :

Angle Subtended by an Arc of a Circle :

In figure, the angle subtended by the minor arc PQ at O is ∠POQ and the angle subtended by the major arc PQ at O is reflex angle ∠POQ.