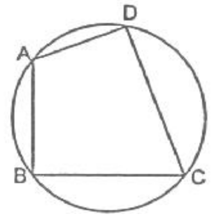
10.4 CYCLIC QUADRILATERAL

A quadrilateral ABCD is called cyclic if all the four vertices of it lie on a circle.

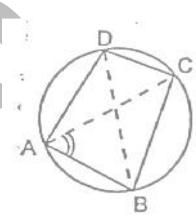


Theorem-7 : The sum of either pair of opposite angles of a cyclic quadrilateral is 180^0

Given : A cyclic quadrilateral ABCD.

To Prove : $\angle A + \angle C = \angle B + \angle D = 180^0$

Construction : Join AC and BD.



Proof : $\angle ACB = \angle ADB$ [Angles of same segment]

And $\angle BAC = \angle BDC$ [Angles of same segment]

$\therefore$ $\angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle ADC$.

Adding $\angle ABC$ to both sides, we get

$\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$.

The left side being the sum of three angles of ΔABC is equal to 180^0.

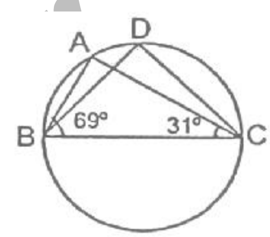
$\therefore$ $\angle ADC + \angle ABC = 180^0$

i.e., $\angle D + \angle B = 180^0$

$\therefore$ $\angle A + \angle C = 360^0 - (\angle B + \angle D) = 180^0$ [$\therefore \angle A + \angle B + \angle C + \angle D = 360^0$] **Hence Proved.**

Corollary : If the sum of a pair of opposite angles of a quadrilateral is 180^0, then quadrilateral is cyclic.

Ex.1 In figure, $\angle ABC = 69^0$, $\angle ACB = 31^0$, find $\angle BDC$.



Sol. In ΔABC.

$\angle BAC + \angle ABC + \angle ACB = 180^0$

[Sum of all the angles of a triangle is 180^0]

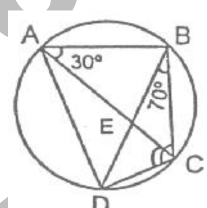
$\Rightarrow$ $\angle BAC + 69^0 + 31^0 = 180^0$

$\Rightarrow$ $\angle BAC + 100^0 = 180^0$

$\Rightarrow$ $\angle BAC = 180^0 - 100^0 = 80^0$

Now, $\angle BDC = \angle BAC = 80^0$. **Ans.** [Angles in the same segment of a circle are equal]

Ex.2 ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70^0$, $\angle BAC$ is 30^0, find $\angle BCD$. Further, if B = BC, find $\angle ECD$.



Sol. $\angle CDB = \angle BAC = 30^0$ ...(i) [Angles in the same segment of a circle are equal]

$\angle DBC = 70^0$ ....(ii)

In ΔBCD,

$\angle BCD + \angle DBC + \angle CDB = 180^0$ [Sum of all he angles of a triangle is 180^0]

$\Rightarrow$ $\angle BCD + 70^0 + 30.0 = 180^0$ [Using (i) and (ii)]

$\Rightarrow$ $\angle BCD + 100^0 = 180^0$

$\Rightarrow \quad \angle BCD = 180^0 - 100^0$

$\Rightarrow \quad \angle BCD = 80^0 \qquad$ ...(iii)

In ΔABC,

$AB = BC$

$\therefore \quad \angle BCA = \angle BAC = 30^0 \qquad$ ...(iv) [Angles opposite to equal sides of a triangle are equal]

Now, $\angle BCD = 80^0 \qquad$ [From (iii)]

$\Rightarrow \quad \angle BCA + \angle ECD = 80^0$

$\Rightarrow \quad 30^0 + \angle ECD = 80^0$

$\Rightarrow \quad \angle ECD = 80^0 - 30^0$

$\Rightarrow \quad \angle ECD = 50^0$

Ex.3 If the nonparallel side of a trapezium are equal, prove that it is cyclic.

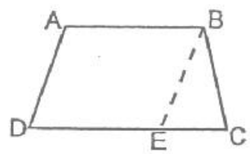
Sol. **Given :** ABCD is a trapezium whose two non-parallel sides AB and BC are equal.

To Prove : Trapezium ABCD is a cyclic.

Construction : Draw BE ∥ AD.

Proof : $\therefore$ AB ∥ DE [Given]

AD ∥ BE [By construction]



$\therefore$ Quadrilateral ABCD is a parallelogram.

$\therefore \quad \angle BAD = \angle BED \qquad$ ....(i) [Opp. angles of a ∥gm]

And, $AD = BE \qquad$ ....(ii) [Opp. sides of a ∥gm]

But $AD = BC \qquad$ ...(iii) [Given]

From (ii) and (iii),

$BE = BC$

$\therefore \quad \angle BEC = \angle BCE \qquad$ ....(iv) [Angles opposite to equal sides]

$\angle BEC + \angle BED = 180^0 \qquad$ [Linear Pair Axiom]

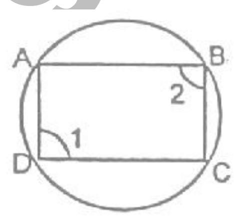
$\Rightarrow \quad \angle BCE + \angle BAD = 180^0 \qquad$ [From (iv) and (i)]

$\Rightarrow$ Trapezium ABCD is cyclic.

[$\therefore$ If a pair of opposite angles of a quadrilateral 180^0, then the quadrilateral is cyclic] **Hence Proved.**

Ex.4 Prove that a cyclic parallelogram is a rectangle.

Sol. **Given :** ABCD is a cyclic parallelogram.



To Prove : ABCD is a rectangle.

Proof : $\therefore$ ABCD is a cyclic quadrilateral

$\therefore \quad \angle 1 + \angle 2 = 180^0 \qquad$ ....(i)

[$\therefore$ Opposite angles of a cyclic quadrilateral are supplementary]

$\therefore$ ABCD is a parallelogram

$\therefore \quad \angle 1 = \angle 2 \qquad$ ...(ii) [Opp. angles of a ∥ gm]

From (i) and (ii),

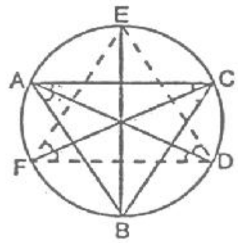
$\angle 1 = \angle 2 = 90^0$

$\therefore$ ∥gm ABCD is a rectangle. **Hence Proved.**

Ex.5 Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively.

Prove that the angles of the triangle DEF are $90^0 - \frac{1}{2}A, 90^0 - \frac{1}{2}B \text{ and } 90^0 - \frac{\angle C}{2}$.

Sol. **Given :** Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively.



To Prove : The angles of the ΔDEF are $90^0 - \frac{\angle A}{2}, 90^0 - \frac{\angle B}{2} \text{ and } 90^0 - \frac{C}{2}$ respectively.

Construction : Join DE, EF and FD.

Proof : $\angle FDE = \angle FDA + \angle EDA = \angle FCA + \angle EBA$ [$\therefore$ Angles in the same segment are equal]

$= \frac{1}{2}\angle C + \frac{1}{2}\angle B$

$\Rightarrow \quad \angle D = \frac{\angle C + \angle B}{2} = \frac{180^0 - \angle A}{2}$ [$\therefore$ In ΔABC, $\angle A + \angle B + \angle C = 180^0$]

$\Rightarrow \quad \angle D = 90^0 - \frac{\angle A}{2}$

Similarly, we can show that

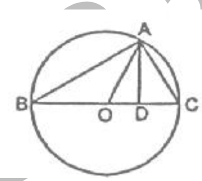
$\angle E = 90^0 - \frac{\angle B}{2}$

And $\quad \angle F = 90^0 - \frac{\angle C}{2}$. **Hence Proved.**

Ex.6 Find the area of a triangle, the radius of whose circumcircle is 3 cm and the length of the altitude drawn from the opposite vertex to the hypotenuse is 2 cm.

Sol. We know that the hypotenuse of a right angled triangle is the diameter of its circumcircle.

$\therefore \quad BC = 2OB = 2 \times 3 = 6$ cm



Let, AD $\perp$ BC

AD = 2 cm [Given]

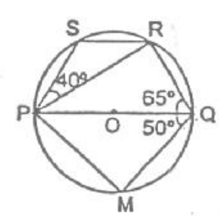
$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}(BC)(AD)$

$= \frac{1}{2}(6)(2)$

$= 6\ cm^2$. **Ans.**

Ex.7 In figure, PQ is a diameter of a circle with centre O. IF $\angle PQR = 65^0$, $\angle SPR = 40^0$, $\angle PQ M = 50^0$, find $\angle QPR$, $\angle PRS$ and $\angle QPM$.

Sol. (i) $\angle QPR$



$\therefore$ PQ is a diameter

$\therefore \quad \angle PRQ = 90^0$ [Angle in a semi-circle is 90^0]

In ΔPQR,

$\angle QPR + \angle PRQ + \angle PQR = 180^0$ [Angle Sum Property of a triangle]

$\Rightarrow \quad \angle QPR + 90^0 + 65^0 = 180^0$

$\Rightarrow \quad \angle QPR + 155^0 = 180^0$

$\Rightarrow$ $\angle QPR = 180^0 - 155^0$

$\Rightarrow$ $\angle QPR = 25^0$.

(ii) $\angle PRS$

$\therefore$ PQRS is a cyclic quadrilateral

$\therefore$ $\angle PSR + \angle PQR = 180^0$ [$\therefore$ Opposite angles of a cyclic quadrilateral are supplementary]

$\Rightarrow$ $\angle PSR + 65^0 = 180^0$

$\Rightarrow$ $\angle PSR = 180^0 - 65^0$

$\Rightarrow$ $\angle PSR = 115^0$

In ΔPSR,

$\angle PSR + \angle SPR + \angle PRS = 180^0$ [Angles Sum Property of a triangle]

$\Rightarrow$ $115^0 + 40^0 + \angle PRS = 180^0$

$\Rightarrow$ $115^0 + \angle PRS = 180^0$

$\Rightarrow$ $\angle PRS = 180^0 - 155^0$

$\Rightarrow$ $\angle PRS = 25^0$

(iii) $\angle QPM$

$\therefore$ PQ is a diameter

$\therefore$ $\angle PMQ = 90^0$ [$\because$ Angle in a semi - circle is 90^0]

In ΔPMQ,

$\angle PMQ + \angle PQM + \angle QPM = 180^0$ [Angle sum Property of a triangle]

$\Rightarrow$ $90^0 + 50^0 + \angle QPM = 180^0$

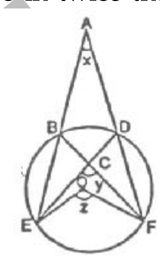
$\Rightarrow$ $140^0 + \angle QPM = 180^0$

$\Rightarrow$ $\angle QPM = 180^0 - 140^0$

$\Rightarrow$ $\angle QPM = 40^0$.

Ex.8 In figure, O is the centre of the circle. Prove that $\angle x + \angle y = \angle z$.

Sol. $\angle EBF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$ [$\because$ Angle subtended by an arc of a circle at the centre in twice the angle subtended by it at any point of the remaining part of the circle]



$\therefore$ $\angle ABF = 180^0 - \frac{1}{2}\angle z$ ...(i) [Linear Pair Axiom]

$\angle EDF = \frac{1}{2}\angle EOF = \frac{1}{2}\angle z$

[$\because$ Angle subtend by any arc of a circle at the centre is twice the angle subtended by it at any point of the remaining part of the circle]

$\therefore$ $\angle ADE = 180^0 - \frac{1}{2}\angle z$ ....(ii) [Linear Pair Axiom]

$\angle BCD = \angle ECF = \angle y$ [Vert. Opp. Angle]

$\angle BAD = \angle x$

In quadrilateral ABCD

$\angle ABC + \angle BCD + \angle CDA + \angle BAD = 2 \times 180^0$ [Angle Sum Property of a quadrilateral]

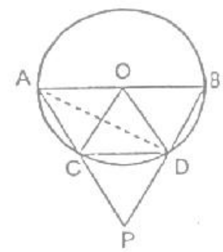
$\Rightarrow \quad 180^0 - \frac{1}{2}\angle z + \angle y + 180^0 - \frac{1}{2}\angle z + \angle x = 2 \times 180^0$

$\Rightarrow \quad \angle x + \angle y = \angle z$ **Hence Proved.**

Ex.9 AB is a diameter of the circle with centre O and chord CD is equal to radius OC, AC and BD produced meet at P. Prove that $\angle CPD = 60^0$.

Sol. **Given :** AB is a diameter of the circle with centre O and chord CD is equal to radius OC. AC and BD produced meet at P.

To Prove : $\angle CPD = 60^0$



Construction : Join AD.

Proof : In ΔOCD,

$OC = OD$ ...(i) [Radii of the same circle]

$OC = CD$ ....(ii) [Given]

From (i) and (ii),

$OC = OD = CD$

$\therefore$ ΔOCD is equilateral

$\therefore$ $\angle COD = 60^0$

$\therefore$ $\angle CAD = \frac{1}{2}\angle COD = \frac{1}{2}\angle(60^0) = 30^0$

[$\because$ Angle subtended by any arc of a circle at the centre is twice the angle subtended by it at any point of the reaming part of the circle]

$\Rightarrow \quad \angle PAD = 30^0$ .....(iii)

And, $\angle ADB = 90^0$ .....(iv) [Angle in a semi-circle]

$\Rightarrow \quad \angle ADB + \angle ADP = 180^0$ [Linear Pair Axiom]

$\Rightarrow \quad 90^0 + \angle ADP = 180^0$ [From (iv)]

$\Rightarrow \quad \angle ADP = 90^0$ ....(v)

In ΔDP,

$\angle ADP + \angle PAD + \angle ADP = 180^0$ [$\because$ The sum of the three angles of a triangles is 180^0]

$\Rightarrow \quad \angle APD + 30^0 + 90^0 = 180^0$ [From (iii) and (v)]

$\Rightarrow \quad \angle APD + 120^0 = 180^0$

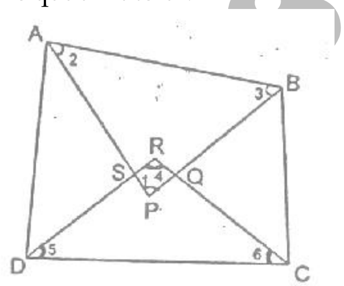
$\Rightarrow \quad \angle APD = 180^0 - 120^0 = 60^0$

$\Rightarrow \quad \angle CPD = 60^0$. **Hence Proved.**

Ex.10 Prove that the quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.

Sol. **Given :** ABCD is a cyclic quadrilateral. Its angle bisectors from a quadrilateral PQRS.

To Prove : PQRS is a cyclic quadrilateral.



Proof : $\angle 1 + \angle 2 + \angle 3 = 180^0$ ...(i) [$\because$ Sum of the angles of a Δ is 180^0]

$\angle 4 + \angle 5 + \angle 6 = 180^0$ ...(ii) [$\because$ Su m of the angles of a Δ is 180^0]

$\therefore$ $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 360$.. ....(iii) [Adding (i) and (ii)]

But $\angle 2 + \angle 3 + \angle 6 + \angle 5 = \frac{1}{2}[\angle A + \angle B + \angle C + \angle D]$

$= \frac{1}{2} . 360^0 = 180^0$ [$\because$ Sum of the angles of quadrilateral is 360^0]

$\therefore$ $\angle 1 + \angle 4 = 360^0 - (\angle 2 + \angle 3 + \angle 6 + \angle 5)$

$\therefore$ PQRS is a cyclic quadrilateral.

[$\because$ If the sum of any pair of opposite angles of a quadrilateral is 180^0, then the quadrilateral is a cyclic]

Hence Proved.

Ex.11 Prove that the angle bisectors of the angles formed by producing opposite sides of a cyclic quadrilateral (Provided they are not parallel) intersect a right angle.

Sol. **Given :** ABCD is a cyclic quadrilateral. Its opposite sides DA and CB are produced to meet at P and opposite sides AB and DC are produced to meet at Q. The bisectors of $\angle P$ and $\angle Q$ meet is F.

To Prove : $\angle PFQ = 90^0$.

Construction : Produce PF to meet DC is G.

Proof : In ΔPEB,

$\angle 5 = \angle 2 + \angle 6$ .....(i)

[$\because$ Exterior angle of a triangle is equal to the sum of interior opposite angles]

But $\angle 2 = \angle 1$

And, $\angle 6 = \angle D$ [$\because$ In a cyclic quadrilateral, exterior angle = interior opposite angle]

$\therefore$ $\angle 5 = \angle 1 + \angle D$ ....(ii) [From (i)]

Now in ΔPDG,

$\angle 7 = \angle 1 + \angle D$ ...(iii)

[$\because$ Exterior angle of a triangle is equal to the sum of interior opposite angles]

Frim (ii) and (iii), we have

$\angle 5 = \angle 7$

Now, in Δ QEF and Δ QGF, [Proved above]

$\angle 5 = \angle 7$ [Common side]

QF = QF [Given]

$\angle 3 = \angle 4$ [AAS criterion]

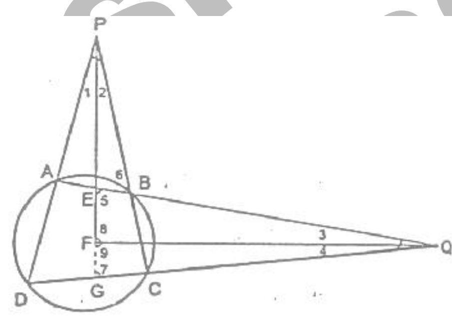
$\therefore$ $\Delta QEF \cong \Delta QGE$ [By cpctc]

$\therefore$ $\angle 8 = \angle 9$

But $\angle 8 + \angle 9 = 180^0$

$\therefore$ $\angle 8 = \angle 9 = 90^0$ [Linear Pair Axiom]

$\therefore$ $\angle PFQ = 90^0$



Hence Proved.

Ex.12 Two concentric circles with centre O have A, B, C, D as the points of intersection with the line ℓ as shown in the figure. If AD = 12 cm and BC = 8 cm, find the length of AB, CD, AC and BD.

Sol. Since OM $\perp$ BC, a chord of the circle,

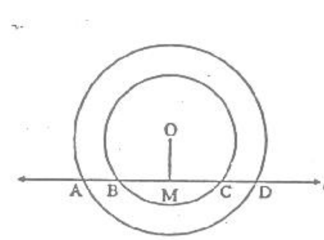
$\therefore$ is bisects BC.

$\therefore$ $BM = CM = \frac{1}{2}(BC) = \frac{1}{2}(8) = 4$ cm

Since, OM $\perp$ AD, a chord of the circle,

$\therefore$ it bisects AD.

$\therefore$ $AM = AD = \frac{1}{2} AD = \frac{1}{2}(8) = 4$ cm



Since, OM $\perp$ CD, a chord of the circle,

$\therefore$ it bisects AD.

$\therefore$ AM = MD = $\frac{1}{2}$ AD = $\frac{1}{2}$ (12) = 6 cm

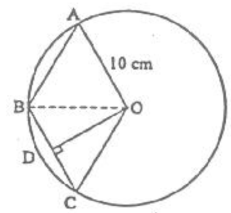
Now, AB = AM - BM = 6 - 4 = 2 cm

CD = MD - MD = 6 - 4 = 2 cm

AC = AM + MC = 6 + 4 = 10 cm

BD = BM + MD = 4 + 6 = 10 cm

Ex.13 OABC is a rhombus whose three vertices, A B and C lie on a circle with centre O. If the radius of the circle is 10 cm. Find the area of the rhombus.



Sol. Since OABC is a rhombus

$\therefore$ OA = AB = BC = OC = 10 cm

Now, OD $\perp$ BC $\Rightarrow$ CD = $\frac{1}{2}$ BC = $\frac{1}{2}$ (10) = 5 cm

$\therefore$ By Pythagoras theorem,

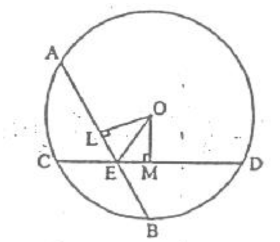
$OC^2 = OD^2 + DC^2$

$\Rightarrow$ $OD^2 = OC^2 - DC^2 = (10)^2 - (5)^2 = 100 - 25 = 75$

$\Rightarrow$ OD = $\sqrt{75} = 5\sqrt{3}$

$\therefore$ Area (ΔOBC) = $\frac{1}{2}$ BC × OD = $\frac{1}{2}$ (10) × $5\sqrt{3} = 25\sqrt{3}$ sq. cm.

Ex.14 Chords AB and CD of a circle with centre O, intersect at a point E. If OE objects $\angle$AED. Prove that AB = CD.



Sol. In ΔOLE and ΔOME

$\angle$OLE = $\angle$OME [90^0 each]

$\angle$LEO = $\angle$MEO [Given]

And OE = OE [Common]

$\therefore$ $\Delta OLE \cong \Delta OME$ [By AAS Criteria]

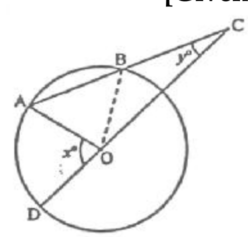
$\Rightarrow$ OL = OM [By cpctc]

This chords AB and CD are equidistant from centre. But we know that only equal chords are equidistant from centre.

$\Rightarrow$ AB = DC

Ex.15 In the given figure. AB is the chord of a circle with centre O. AB is produced to C such that BC = OB. CO is joined and produced to meet the circle in D. If $\angle ACD = y^0$ and $\angle AOD = x^0$, prove that $x^0 = 3y^0$.

Sol. Since BC = OB **[Given]**



$\therefore$ $\angle OCB = \angle BOC = y^0$ [$\because$ Angles opposite to equal sides are equal]

$\angle OBA = \angle BOC + \angle OCB = y^0 + y^0 = 2y^0$.

[$\because$ Exterior angle of a Δ is equal to the sum of the opposite interior angles]

Also OA = OB [Radii of the same circle]

$\angle OAB = \angle OBA = 2y^0$ [Angles opposite to equal sides of a triangle are equal]

$\angle AOD = \angle OAC + \angle OCA$

$= 2y^0 + y^0 = 3y^0$

[$\because$ Exterior angle of a Δ is equal to the sum of the opposite interior angles]

Hence $x^0 = 3y^0$ **Hence Proved.**

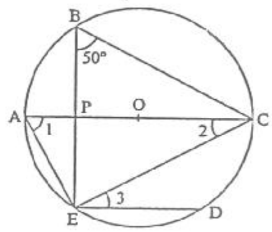
Ex.16 In the given figure, the chord ED is parallel to the diameter AC. Find $\angle CED$.

Sol. $\angle CBE = \angle 1$ [$\angle$s in the same segment]

$\angle 1 = 50^0$ ....(i) [$\because \angle CBE = 50^0$]

$\angle AEC = 90^0$ ....(ii) [Semicircle Angle is right angle]

Now, in ΔAEC,



$\angle 1 + \angle AEC + \angle 2 = 180^0$ [$\because$ Sum of angles of a $\Delta = 180^0$]

$\therefore$ $50^0 + 90^0 + \angle 2 = 180^0$

$\Rightarrow$ $\angle 2 = 180^0 - 140^0 = 40^0$

Thus $\angle 2 = 40^0$ ....(iii)

Also, ED $\parallel$ AC [Given]

$\therefore$ $\angle @ = \angle 3$ [Alternate angles]

$\therefore$ $40^0 = \angle 3$ i.e., $\angle 3 = 40^0$

Hence $\angle CED = 40^0$ **Ans.**

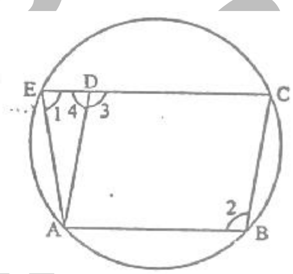
Ex.17 ABCD is a parallelogram. The circle through A, B, C intersects CD (produced if necessary) at E. Prove that AD = AE.

Sol. **Given :** ABCD is a parallelogram. The circle through A, B, C intersects CD, when produced in E.

To prove : AE = AD.

Proof : Since ABCE is a cyclic quadrilateral

$\therefore$ $\angle 1 + \angle 2 = 180^0$ ....(i) [opposite angles of a cyclio quadrilateral are supplementary]



Also $\angle 3 + \angle 4 = 180^0$ [linear pair] ....(ii)

From (i) and (ii), we get $\angle 1 + \angle 2 = \angle 3 + \angle 4$ ....(iii)

But $\angle 2 = \angle 3$ ....(iv)

$\therefore$ From (iii) and (iv), we get $\angle 1 = \angle 4$

Now in ΔADE, since $\angle 1 = \angle 4$

AD = AE [Sides opp. to equal angles of a triangle are equal]

Hence Proved.