2.2 ALGEBRAIC IDENTITYAND THEIR APPLICATIONS

An identity is an equality which is true for all values of the variables

Some important identities are

(i) $(a + b)^2 = a^2 + 2ab + b^2$

(ii) $(a - b)^2 = a^2 - 2ab + b^2$

(iii) $a^2 - b^2 = (a + b)(a - b)$

(iv) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

(v) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

(vi) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(vii) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(viii) $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$

(ix) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Special case : if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$.

(a) Value Form :

(i) $a^2 + b^2 = (a + b)^2 - 2ab$, if a + b and ab are given

(ii) $a^2 + b^2 = (a - b)^2 + 2ab$ if a - b and ab are given

(iii) $a + b = \sqrt{(a-b)^2 + 4ab}$ if a - b and ab are given

(iv) $a - b = \sqrt{(a+b)^2 - 4ab}$ if a + b and ab are given

(v) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$ if $a + \frac{1}{a}$ is given

(vi) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 + 2$ if $a - \frac{1}{a}$ is given

(vii) $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ if (a + b) and ab are given

(viii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$ if (a - b) and ab are given

(ix) $x^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right)$ if $a + \frac{1}{a}$ is given

(x) $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 + 3\left(a - 1\frac{1}{a}\right)$, if $\left(a - \frac{1}{a}\right)$ is given

(xi) $a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = [(a + b)^2 - 2ab]^2 - 2a^2b^2$, if (a + b) and ab are given

(xii) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = [(a + b)^2 - 2ab](a + b)(a - b)$

(xiii) $a^5 + b^5 = (a^3 + b^3)(a^2 + b^2) - a^2b^2(a + b)$

ILLUSTRATION

Ex.1 Find the value of :

(i) $36x^2 + 49y^2 + 84xy$, when x = 3, y = 6

(ii) $25x^2 + 16y^2 - 40xy$, when x = 6, y = 7

Sol. (i) $36x^2 + 49y^2 + 84xy = (6x)^2 + (7y)^2 + 2 \times (6x) \times (7y)$

$= (6x + 7y)^2$

$= (6 \times 3 + 7 \times 6)^2$ [When x = 3, y = 6]

$= (18 + 42)^2$

$= (60)^2$

$= 3600$. **Ans.**

(ii) $25x^2 + 16y^2 - 40xy = (5x)^2 + (4y)^2 - 2 \times (5x) \times (4y)$

$= (5x - 4y)^2$

$= (5 \times 6 - 4 \times 7)^2$ [When x = 6, y = 7]

$= (30 - 28)^2$

$= 2^2$

$= 4$

Ans.

Ex.2 If $x^2 + \frac{1}{x^2} = 23$, find the value of $\left(x + \frac{1}{x}\right)$.

Sol. $x^2 + \frac{1}{x^2} = 23$ ....(i)

$\Rightarrow$ $x^2 + \frac{1}{x^2} + 2 = 25$ [Adding 2 on both sides of (i)]

$\Rightarrow$ $(x^2) + \left(\frac{1}{x}\right)^2 + 2 . x . \frac{1}{x} = 25$

$\Rightarrow$ $\left(x + \frac{1}{x}\right)^2 = (5)^2$

$\Rightarrow$ $x + \frac{1}{x} = 5$ **Ans.**

Ex.3 Prove that $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$.

Sol. Here, L.H.S. $= a^2 + b^2 + c^2 - ab -+ bc - ca$

$= \frac{1}{2}\ [2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$

$= \frac{1}{2}\,[(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)]$

$= \frac{1}{2}\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$

= RHS Hence Proved.

Ex.4 **Evaluate :**

(i) $(107)^2$ (ii) $(94)^2$ (iii) $(0.99)^2$

Sol. (i) $(107)^2 = (100 + 7)^2$

$= (100)^2 + (7)^2 + 2 \times 100 \times 7$

$= 10000 + 49 + 1400$

$= 11449$ **Ans.**

(ii) $(94)^2 = (100 - 6)^2$

$= (100)^2 + (6)^2 - 2 \times 100 \times 6$

$= 10000 + 36 - 1200$

$= 8836$ **Ans.**

(iii) $(0.99)^2 = (1 - 0.01)^2$

$= (1)^2 + (0.01)^2 - 2 \times 1 \times 0.01$

$= \ + 0.0001 - 0.02$

$= 0.9801$ **Ans.**

NOTE : We may extend the formula for squaring a binomial to the squaring of a trinomial as given below.

$(a + b + c)^2 = [a + (b + c)]^2$

$= a^2 + (b + c)^2 + 2 \times a \times (b + c)$ [Using the identity for the square of binomial]

$= a^2 + b^2 + c^2 + 2bc + 2\ (b + c)$ [Using $(b + c)^2 = b^2 + c^2 + 2bc$]

$= a^2 + b^2 + c^2 + 2bc + 2ab + 2ac$ [Using the distributive law]

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

$\therefore\ (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

Ex.5 Simplify : $(3x + 4)^3 - (3x - 4)^3$.

Sol. We have,

$(3x + 4)^3 - (3x - 4)^3 = [(3x)^3 + (4)^3 + 3 \times 3x \times 4 \times (3x + 4)] - [(3x)^3 - (4)^3 - 3 \times 3x \times 4 \times (3x - 4)]$

$= [27^3 + 64 + 36x\ (3x + 4)] - [27^3 - 64 - 36x\ (3x - 4)]$

$= [27x^3 + 64 + 108x^2 + 144x] - [27x^3 - 64 - 108x^2 + 144x]$

$= 27x^3 + 64 + 108x^2 + 144x - 27x^3 + 64 + 108x^2 - 144x$

$= 128 + 216x^2$

$\because\ (3x + 4)^3 - (3x - 4)^3 = 128 + 216x^2$ **Ans.**

Ex.6 Evaluate :

(i) $(1005)^3$ (ii) $(997)^3$

Sol. (i) $(1005)^3 = (1000 + 5)^3$

$= (1000)^3 + (5)^3 + 3 \times 1000 \times 5 \times (1000 + 5)$

$= 1000000000 + 125 + 15000 + (1000 + 5)$

$= 1000000000 + 125 + 15000000 + 75000$

$= 1015075125.$ **Ans.**

(ii) $(997)^3 = (1000 - 3)^3$

$= (1000)^3 - (3)^3 - 3 \times 1000 \times 3 \times (1000 - 3)$

$= 1000000000 - 27 - 9000 \times (1000 - 3)$

$= 1000000000 - 27 - 900000 + 27000$

$= 991026973$ **Ans.**

Ex.7 If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$

Sol. We have, $x - \frac{1}{x} = 5$ ...(i)

$\Rightarrow \left(x - \frac{1}{x}^3 = (5)^3\right)$ [Cubing both sides of (i)]

$\Rightarrow x^3 - \frac{1}{x^3} - 3x.\frac{1}{x}.\left(x - \frac{1}{x}\right) = 125$

$\Rightarrow x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 125$

$\Rightarrow x^3 - \frac{1}{x^3} - 3 \times 5 = 125$ [Substituting $\left(x - \frac{1}{x}\right) = 5$]

$\Rightarrow x^3 - \frac{1}{x^3} - 15 = 125$

$\Rightarrow x^3 - \frac{1}{x^3} = (125 + 15) = 140$ **Ans.**

Ex.8 Find the following products of the following expression :

(i) $(4x + 3y)(16x^2 - 12xy + 9y^2)$ (ii) $(5x - 2y)(25x^2 + 10xy + 4y^2)$

Sol. (i) $(4x + 3y)(16x^2 - 12xy + 9y^2)$

$= (4x + 3y)[(4x)^2 - (4x) \times (3y) + (3y)^2]$

$= (x + b)(x^2 - ab + b^2)$ [Where a = 4x, b = 3y]

$= a^3 + b^3$

$= (4x)^3 + (3y)^3 = 64x^3 + 27y^3$ **Ans.**

(ii) $(5x - 2y)(25x^2 + 10xy + 4y^2)$

$= (5x - 2y)[(5x^2 + (5x) \times (2y) + (2y)^2]$

$= (a - b)(a^2 + ab + b^2)$ [Where a = 5x, b = 2y]

$= a^3 - b^3$

$= (5x)^3 - (2y)^3$

$= 125x^3 - 8y^3$ **Ans.**

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Ex.9 Simplify : $\frac{(a^2-b^2)^3+(v^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

Sol. Here $(a^2-b^2)+(b^2-c^2)^3+(c^2-a^2)=0$

$\therefore \quad (a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3 = 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$

Also, $(a-b)+(b-c)+(c-a)=0$

$\therefore \quad (a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a)$

$\therefore$ Given expression $= \frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$

$= \frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$

$=(a+b)(b+c)(c+a)$ **Ans.**

Ex.10 Prove that : $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

Sol. Let $(x - y) = a$, $(y - z) = b$ and $(z - x) = c$.

Then, $a + b + c = (x - y) + (y - z) + (z - x) = 0$

$\therefore a^3 + b^3 + c^3 = 3abc$

Or $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$ **Ans.**

Ex.11 Find the value of $(28)^3 - (78)^3 + (50)^3$.

Sol. Let $a = 28$, $b = -78$, $c = 50$

Then, $a + b + c = 28 - 78 + 50 = 0$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$.

So, $(28)^3 + (-78)^3 + (50)^3 = 3 \times 28 \times (-78) \times 50$

Ex.12 If $a + b + c = 9$ and $ab + bc + ac = 26$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Sol. We have $a + b + c = 9$ ...(i)

$\Rightarrow \quad (a + b + c)^2 = 81$ [On squaring both sides of (i)]

$\Rightarrow \quad a^2 + b^2 + c^2 + 2(ab + bc + ac) = 81$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2 \times 26 = 81$ [$\because ab + bc + ac = 26$]

$\Rightarrow \quad a^2 + b^2 + c^2 = (81 - 52)$

$\Rightarrow \quad a^2 + b^2 + {}^2 = 29$.

Now, we have

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

$= (a + b + c)[(a^2 + b^2 + c^2) - (ab + bc + ac)]$

$= 9 \times [(29 - 26)]$

$= (9 \times 3)$

$= 27$ **Ans.**

(b) A Special Product :

We have $(x + a)(x + b) = x(x + b) + a(x + b)$

$= x^2 + xb + ax + ab$

$= x^2 + bx + ax + ab$ $\quad [\because xb = bx]$

$= x^2 + ax + bx + ab$

$= x^2 + (a + b)x + ab$

Thus, we have the following identity

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Ex.13 Find the following products :

(i) $(x + 2)(x + 3)$ (ii) $(x + 7)(x - 2)$

(ii) $(y - 4)(y + 3)$ (iv) $(y - 7)(y + 3)$

(v) $(2x - 3)(2x - 5)$ (vi) $(3x + 4)(3x - 5)$

Sol. Using the identity : $(x + a)(x + b) = x^2 + (a + b)x + ab$, we have

(i) $(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \times 3$

$= x^2 + 5x + 6.$ **Ans.**

(ii) $(x + 7)(x - 2) = (x + 7)(x + (-2))$

$= x^2 + 7x + (-2)x + 7 \times (-2)$

$= x^2 + 5x - 14.$ **Ans.**

(iii) $(y - 4)(y - 3) = \{y + (-4)\}\{y + (-3)\}$

$= y^2 + \{(-4) + (-3)\}y + (-4) \times (-3)$

$= y^2 - 7y + 12$ **Ans.**

(iv) $(y - 7)(y + 3) = \{y + (-7)\}(y + 3)$

$= y^2 + \{(-7) + 3\} + (-7) \times 3$

$= y^2 - 4y - 21.$ **Ans.**

(v) $(2x - 3)(2x + 5) = (y - 3)(y + 5)$, where $y = 2x$

$= \{y + (-3)\}(y + 5)$

$= y^2 + \{(-3) + 5\}y + (-3) \times 5$

$= y^2 + 2y - 15$

$= (2x)^2 + 2 \times 2x - 15$

$= 4x^2 + 4x - 15.$ **Ans.**

(vi) $(3x + 4)(3x - 5) = (y + 4)(y - 5)$, where $y = 3x$

$= (y + 4)\{y + (-5)\}$

$= y^2 + \{4 + (-5)\} + 4 \times (-5)$

$= y^2 - y - 20$

$= (3x)^2 - 3x - 20$

$= 9x^2 - 3x - 20.$ **Ans.**

Ex.14 Evaluate : (i) 35×37 (ii) 103×96

Sol. (i) $35 \times 37 = (40 - 5)(40 - 3)$

$= (40 + (-5))(40 + (-3))$

$= 40^2 + (-5 - 3) \times 40 + (-5 \times -3)$

$= 1600 - 320 + 15$

$= 1615 - 320$

$= 1295$ **Ans.**

(ii) 103×96

$= (100 + 3)\,[100 + (-4)]$

$= 100^2 + (3 + (-4)) \times 100 + (3 \times -4)$

$= 10000 - 100 - 12$

$= 9888$ **Ans.**