2.5 REMAINDER THEOREM

Let 'p(x)' be any polynomial of degree greater than or equal to one and a be any real number and If p(x) is divided by (x - a), then the remainder is equal to p(a).

Let q(x) be the quotient and r(x) be the remainder when p(x) is divided by (x - a) then

Dividend = Divisor × Quotient + Remainder

p(x) = (x - a) × q(x) + [r(x) or r], where r(x) = 0 or degree of r(x) < degree of (x -). But (x - a) is a polynomial of degree 1 and a polynomial of degree less than 1 is a constant. Therefore, either r(x) = 0 or r(x) = Constant.

Let r(x) = r, then p(x) = (x - a)q(x) + r,

putting x = a in above equation p(a) = (a - a)q(a) + r = 0. q(a) + r

p(a) = 0 + r

$\Rightarrow$ p(a) = r

This shows that the remainder is p(a) when p(x) is divided by (x - a).

REMARK : If a polynomial p(x) is divided by (x + a), (ax - b), (x + b), (b - ax) then the remainder in the value of p(x) at $x = -a, \frac{b}{a}, -\frac{b}{a}, \frac{b}{a}$ i.e. p(-a), $p\left(\frac{b}{a}\right), p\left(-\frac{b}{a}\right), p\left(\frac{b}{a}\right)$ respectively.

Ex.4 Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by g(x) = 1- 2x.

Sol. $1 - 2x = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4$$

$$= \frac{1}{8} - \frac{3}{2} + 1 - 4$$

$$= \frac{1 - 12 + 8 - 32}{8} = -\frac{35}{8}$$ **Ans.**

Ex.5 The polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by x + 2 if the remainder in each case is the same, find the value of a.

Sol. $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$

when p(x) & q(x) are divided by $x + 2 = 0 \Rightarrow x = -2$

p (-2) = q(-2)

$\Rightarrow a(-2)^3 + 3(-2)^2 - 13 = 2(-2)^3 - 5(-2) + a$

$\Rightarrow -8a + 12 - 13 = -16 + 10 + a$

$\Rightarrow -9a = -5$

$\Rightarrow a = \frac{5}{9}$ **Ans.**

(a) Factor Theorem :

Let p(x) be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that p(a) = 0, than (x - a) is a factor of p(x). Conversely, if (x - a) is a factor of p(x), then p(a) = 0.

Ex.6 Show that x + 1 an d 2x - 3 are factors of $2x^3 - 9x^2 + x + 12$.

Sol. To prove that (x + 1) and (2x - 3) are factors of $2x^3 - 9x^2 + x + 12$ it is sufficient to show that p(-1) and $p\left(\frac{3}{2}\right)$ both are equal to zero.

$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12 = -2 - 9 - 1 + 12 = -12 + 12 = 0$

And, $p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right) + 12$

$$= \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{-81 + 81}{4} = 0$$

Hence, (x + 1) and (2x - 3) are the factors $2x^3 - 9x^2 + x + 12$. **Ans.**

Ex.7 Find $\frac{a}{}$ and β if x + 1 and x + 2 are factors of $p(x) = x^3 + 3x^2 - 2\alpha x + \beta$.

Sol. When we put x + 1 = 0 or x = -1 and x + 2 = 0 or x = - 2 in p(x)

Then, p(-1) = 0 & p(-2) = 0

Therefore, $p(-1) = (-1)^3 + 3(-1)^2 - 2\alpha\ (-1) + \beta = 0$

$\Rightarrow\ -1 + 3 + 2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha - 2$ ....(i)

And, $p(-2) = (-2)^3 + 3(-2)^2 - 2\alpha\ (-2) + \beta = 0$

$\Rightarrow\ -8 + 12 + 4\alpha + \beta = 0 \qquad \Rightarrow \beta = -4\alpha - 4$ .....(i)

From equation (i) and (ii)

$-2\alpha - 2 = -4\alpha - 4$

$\Rightarrow\ 2\alpha = -2 \Rightarrow \alpha = -1$

Put $\alpha = -1$ in equation (i) $\Rightarrow \beta = -2(-1) - 2 = 2 - 2 = 0$.

Hence, $\alpha = -1\ \beta = 0$. **Ans.**

Ex.8 What must be added to $3x^3 + x^2 - 22x + 9$ so that the result is exactly divisible by $3x^2 + 7x - 6$.

Sol. Let $p(x) = 3x^3 + x^2 - 22x + 9$ and $q(x) = 3x^2 + 7x - 6$.

We know if p(x) is divided by q(x) which is quadratic polynomial therefore if p(x) is not exactly divisible by q(x) then the remainder be r(x) and degree of r(x) is less than q(x) (or Divisor)

$\therefore$ By long division method

Let we added ax + b (linear polynomial) is p(x), so that p(x) + ax + b is exactly divisible by $3x^2 + 7x - 6$.

Hence $p(X) + ax + b = s(x) = 3x^3 + x^2 - 22x + 9\ ax + b$

$= 3x^3 + x^2 - x(22 - a) + (9 + b)$

$$\begin{array}{r|l} & x - 2 \\ 3x^2 + 7x - 6 & 3x^3 + x^2 - x(22 - a) + 9 + b \\ & -3x^3 + 7x^2 - 6x \\ \hline & -6x^2 + 6x - (22 - a)x + 9 + b \\ & \text{or} \\ & -6x^2 + x(-16 + a) + 9 + b \\ & -6x^2 - 14x \qquad \pm 12 \\ \hline & x(-2 + a) + (b - 3 \end{array}$$

Hence, x(a – 2 + b – 3 = 0 . x + 0)

$\Rightarrow\ a - 2 = 0\ \&\ b - 3 = 0$

$\Rightarrow$ a = 2 or b = 3 **Ans.**

Hence, if we add ax + b or 2x + 3 in p(x) then it is exactly divisible by $3x^2 + 7x - 6$.

Ex.9 Using factor theorem, factories :

$p(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$

Sol. $45 \Rightarrow \pm1, \pm3, \pm5, \pm9, \pm15, \pm45$

if we put x = 1 in p(x)

$p(1) = 2(1)^4 - 7(1)^3 - 13(1)^2 + 63(1) - 45$

$2 - 7 - 13 + 63 - 45 = 65 - 65 = 0$

$\therefore$ x = 1 or x - 1 is a factor of p(x).

Similarly, if we put x = 3 in p(x)

$p(3) = 2(3)^4 - 7(3)^3 - 13(3)^2 + 63(3) - 45$

$162 - 189 - 117 + 189 - 45 = 162 - 162 = 0$

Hence, x = 3 or x - 3 = 0 is the factor of p(x).

$p(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$

$\therefore$ $p(x) = 2x^3(x - 1) -5x^2(x - 1) - 18(x - 1) + 45(x - 1)$

$2x^4 - 2x^3(x - 1) - 5x^2 - 18x^2 + 18x + 45x - 54$

$\Rightarrow$ $p(x) = (x - 1)(2x^3 - 5x^2 - 18x + 45)$

$\Rightarrow$ $p(x) = (x - 1)(2x^3 - 5x^2 - 18x + 45)$

$\Rightarrow$ $p(x) = (x - 1)[2x^2(x - 3) + x(x - 3) - 15(x - 3)]$

$\Rightarrow$ $p(x) = (x - 1)[2x^3 - 6x^2 + x^2 - 3x - 15x + 45]$

$\Rightarrow$ $p(x) = (x - 1)(x - 3)(2x^2 + x - 15)$

$\Rightarrow$ $p(x) = (x - 1)(x - 3)(2x^2 + 6x - 5x - 15)$

$\Rightarrow$ $p(x) = (x - 1)(x - 3)[2x(x + 3) - 5(x + 3)]$

$\Rightarrow$ $p(x) = (x - 1)(x - 3)(x + 3)(2x - 5)$