7.2 CONGRUENT FIGURES

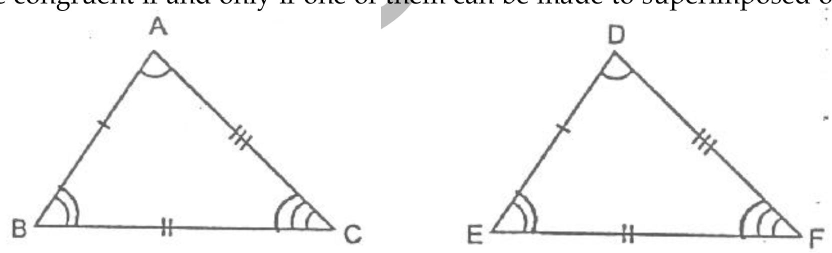
The figures are called congruent if they have same shape and same size. In order words, two figures are called congruent if they are having equal length, width and height.

Fig. (i) Fig. (ii)

In the above figures {fig. (i) and fig. (ii)} both are equal in length, width and height, so these are congruent figures.

(a) Congruent Triangles :

Two triangles are congruent if and only if one of them can be made to superimposed on the other, so an to cover it exactly.



If two triangles ΔABC and ΔDEF are congruent then there exist a one to one correspondence between their vertices and sides. i.e. we get following six equalities.

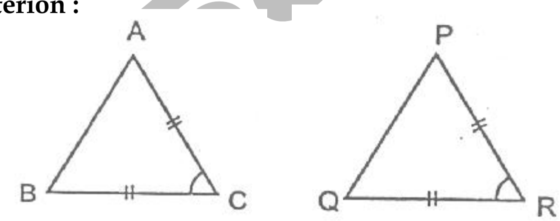
$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and AB = DE, BC = EF, AC = DF.

If two ΔABC & ΔDEF are congruent under $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ one to one correspondence then we write $\Delta ABC \cong \Delta DEF$ we can not write as $\Delta ABC \cong \Delta DFE$ of $\Delta ABC \cong \Delta EDF$ or in other forms because $\Delta ABC \cong \Delta DFE$ have following one-one correspondence $A \leftrightarrow D$, $B \leftrightarrow F$, $C \leftrightarrow E$.

Hence we can say that "two triangles are congruent if and only if there exists a one-one correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal.

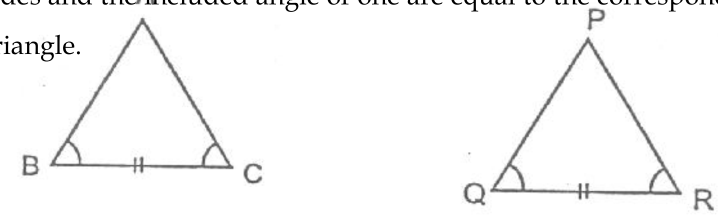
(b) Sufficient Conditions for Congruence of two Triangles :

(i) SAS Congruence Criterion :



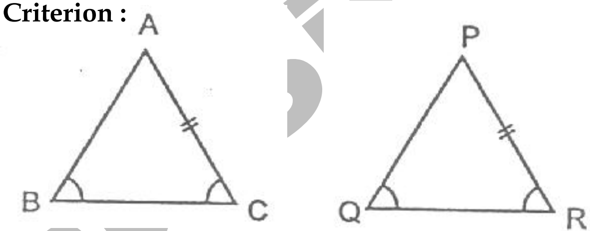
Two triangles are congruent if two sides and the included angle of one are equal to the corresponding sides and the included angle of the other triangle.

(ii) ASA Congruence Criterion :



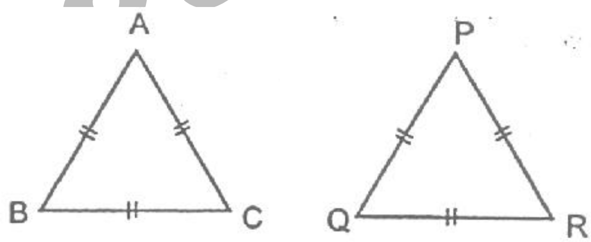
Two triangles are congruent if two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle.

(iii) AAS Congruence Criterion :



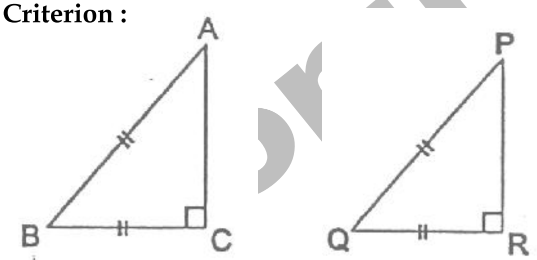
If any two angles and a non included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

(iv) SSS Congruence Criterion :



Two triangles are congruent if the three sides of one triangle are equal to the corresponding three sides of the other triangle.

(v) RHS Congruence Criterion :



Two right triangles are congruent if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other triangle.

(c) Congruence Relation in the Set of all Triangles :

By the definition of congruence of two triangles, we have following results.

(I) Every triangle is congruent to itself i.e. $\Delta ABC \cong \Delta ABC$

(II) If $\Delta ABC \cong \Delta DEF$ then $\Delta DEF \cong \Delta ABC$

(III) If $\Delta ABC \cong \Delta DEF$ and $\Delta DEF \cong \Delta PQR$ then $\Delta ABC \cong \Delta PQR$

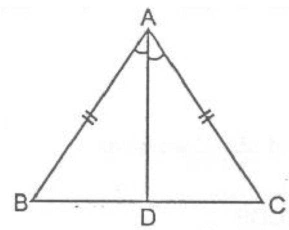
NOTE : If two triangles are congruent then their corresponding sides and angles are also congruent by cpctc (corresponding parts of congruent triangles are also congruent).

Theorem-1 : Angles opposite to equal sides of an isosceles triangle are equal.

Given : ΔABC in which AB = AC

To Prove : ∠B = ∠C

Construction : We draw the bisector AD of ∠A which meets BC in D.



Proof : In ΔABD and ΔACD we have

AB = AC [Given]

∠BAD = ∠CAD [∵ AD is bisector of ∠A]

And, AD = AD [Common side]

∴ By SAS criterion of congruence, we have

ΔABD ≅ ΔACD

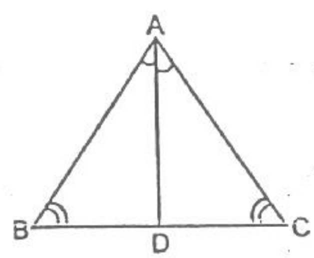
⇒ ∠B = ∠C by cpctc **Hence Proved.**

Theorem - 2: if two angles of a triangle are equal, then sides opposite to them are also equal.

Given : ΔABC in which ∠B = ∠C

To Prove : AB = AC

Construction: We draw the bisector of ∠A which meets BC in D.



Proof : In ΔABD and ΔACD we have

∠B = ∠C [Given]

∠BAD = ∠CAD [∵ AD is bisector of ∠A]

AD = AD [Common side]

∴ By AAS criterion of congruence, we get

ΔABD ≅ ΔACD

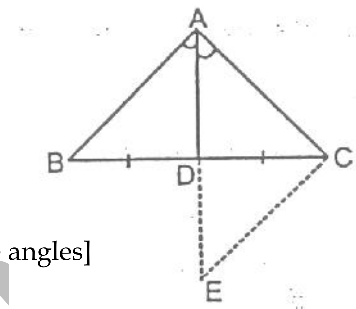
⇒ AB = AC [By cpctc] **Hence, Proved.**

Theorem-3 : if the bisector of the vertical angle bisects the base of the triangle, then the triangle is isosceles.

Given : A ΔABC in which AD is the bisector of ∠A meeting BC in D such that BD = CD

To Prove : ΔACD is an isosceles triangle.

Construction : We produce AD to E such that AD = DE and join EC.



Proof : In ΔADB and ΔEDC we have

$AD = DE$ [By construction]

$\angle ADB = \angle CDE$ [Vertically opposite angles]

$BD = DC$ [Given]

$\therefore$ By SAS criterion of congruence, we get

$\Delta ADB \cong \Delta EDC \Rightarrow AB = EC$ ...(i)

And, $\angle BAD = \angle CED$ [By cpctc]

But, $\angle BAD = \angle CAD$

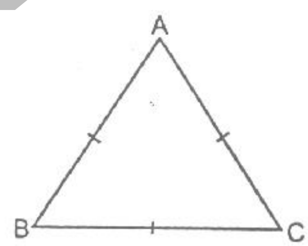
$\therefore$ $\angle CAD = \angle CED$

$\Rightarrow$ $AC = EC$ [Sides opposite to equal angles are equal]

$\Rightarrow$ $AC = AB$ [By eg. (i)] **Hence Proved.**

Ex.1 Prove that measure of each angle of an equilateral triangle is 60^0.

Sol. Let ΔABC be an equilateral triangle, then we have



$AB = BC = CA$ ...(i)

$\therefore$ $AB = BC$

$\therefore$ $\angle C = \angle A$ ...(ii) [Angles opposite to equal sides are equal]

Also, $BC = CA$

$\therefore$ $\angle A = \angle B$ ...(iii) [Angles opposite to equal sides]

By (ii) & (iii) we get $\angle A = \angle B = \angle C$

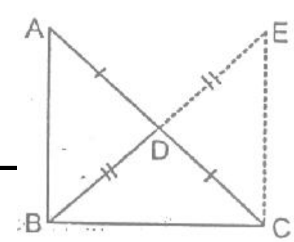
Now in ΔABC $\angle A + \angle B + \angle C = 180^0$

$\Rightarrow$ $3\angle A = 180^0$ [$\therefore \angle A = \angle B = \angle C$]

$\Rightarrow$ $\angle A = 60^0 = \angle B = \angle C$ **Hence Proved.**

Ex.2 If D is the mid-point of the hypotenuse AC of a right triangle ABC, prove that $BD = \frac{1}{2}AC$.

Sol. Let ΔABC is a right triangle such that $\angle B = 90^0$ and D is mid point of AC then we have to prove that $BD = \frac{1}{2}AC$ we produce BD to E such that BD = AC and EC.



Now is ΔADB and ΔCDE we have

$AD = DC$ [Given]

BD = DE [By construction]

And, $\angle ADB = \angle CDE$ [Vertically opposite angles]

$\therefore$ By SAS criterion of congruence we have

$\Delta ADB \cong \Delta CDE$

$\Rightarrow$ $EC = AB$ and $\angle CED = \angle ABD$ ....(i) [By cpctc]

But $\angle CED$ & $\angle ABD$ are alternate interior angles

$\therefore$ $CE \parallel AB \Rightarrow \angle ABC + \angle ECB = 180^0$ [Consecutive interior angles]

$\Rightarrow$ $90 + \angle ECB = 180^0$

$\Rightarrow$ $\angle ECB = 90^0$

Now, In ΔABC & ΔECB we have

$AB = EC$ [By (i)]

$BC = BC$ [Common]

And, $\angle ABC = \angle ECB = 90^0$

$\therefore$ BY SAS criterion of congruence

$\Delta ABC \cong \Delta ECB$

$\Rightarrow$ $AC = EB$ [By cpctc]

$\Rightarrow$ $\frac{1}{2}AC = \frac{1}{2}EB$

$\Rightarrow$ $BD = \frac{1}{2}AC$ **Hence Proved.**

Ex.3 In a right angled triangle, one acute angle is double the other. Prove that the hypotenuse is double the smallest side.

Sol. Let ΔABC is a right triangle such that $\angle B = 90^0$ and $\angle ACB = 2\angle CAB$, then we have to prove $AC = 2BC$.

we produce CB to D such that BD = CB and join AD.

Proof : In ΔABD and ΔABC we have

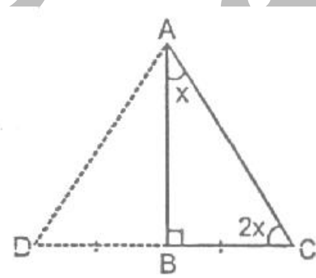
$BD = BC$ [By construction]

$AB = AB$ [Common]

$\angle ABD = \angle ABC = 90^0$

$\therefore$ By SAS criterion of congruence we get

$\Delta ABD \cong \Delta ABC$



$\Rightarrow$ $AD = AC$ and $\angle DAB = \angle CAB$ [By cpctc]

$\Rightarrow$ $AD = AC$ and $\angle DAB = x$ [$\therefore \angle CAB = x$]

Now, $\angle DAC = \angle DAB + \angle CAB = x + x = 2x$

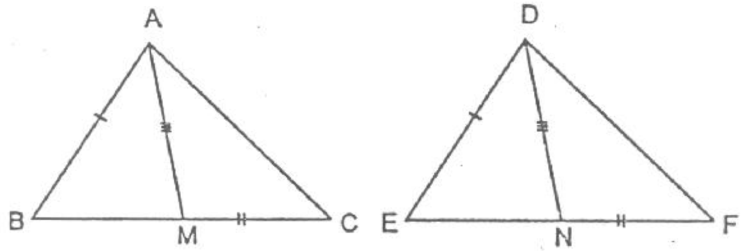
$\therefore$ $\angle DAC = \angle ACD$

$\Rightarrow$ $DC = AD$ [Side Opposite to equal angles]

$\Rightarrow$ $2BC = AD$ [$\because DC = 2BC$]

$\Rightarrow$ $2BC = AC$ [AD = AC] **Hence Proved.**

Ex.4 In figure, two sides AB and BC and the median AM of a ΔABC are respectively equal to sides DE and EF and the median DN of ΔDEF. Prove that $\Delta ABC \cong \Delta DEF$.



Sol. $\because$ AM and DN are medians of ΔABC & ΔDEF respectively

$\therefore$ BM = MC & EN = NF

$\Rightarrow$ BM = $\frac{1}{2}$ BC & EN = $\frac{1}{2}$ EF

But, BC = EF $\therefore$ BM = EN ...(i)

In ΔABM & ΔDEN we have

AB = DE [Given]

AM = DN [Given]

BM = EN [By (i)]

$\therefore$ By SSS criterion of congruence we have

$\Delta ABM \cong \Delta DEN \Rightarrow \angle B = \angle E$ ...(ii) [By cpctc]

Now, In ΔABC & ΔDEF

AB = DE [Given]

$\angle B = \angle E$ [By (ii)]

BC = EF [Given]

$\therefore$ By SAS criterion of congruence we get

$\Delta ABC \cong \Delta DEF$ **Hence Proved.**