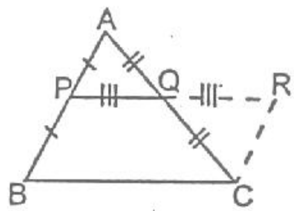
8.3 MID-POINT THEOREM

Statement : In a triangle, the line segment joining the mid-points of any two sides is parallel to the third side and is half of it.



Given : A triangle ABC is which P is the mid-point of side AB and Q is the mid-point of side AC.

To Prove : P is parallel to BC and is half of it i.e., PQ $\parallel$ BC and PQ = $\frac{1}{2}$BC

Construction : Produce PQ upto point R such that PQ = QR. Join T and C.

Proof : In ΔAPQ and ΔCRQ : -

PQ = QR	[By construction]
AQ = QC	[Given]
And, $\angle AQP = \angle CQR$	[Vertically opposite angles]
$\Rightarrow$ $\Delta APQ \cong \Delta CRQ$	[By SAS]
$\Rightarrow$ AP = CR	[By cpctc]
And, $\angle APQ = \angle CRQ$	[By cpctc]

But, $\angle APQ$ and $\angle CRQ$ are alternate angles and we know, whenever the alternate angles are equal, the lines are parallel.

$\Rightarrow$ AP $\parallel$ CR

$\Rightarrow$ AB $\parallel$ CR

$\Rightarrow$ BP $\parallel$ CR

Given, P is mid-point of AB

$\Rightarrow$ AP = BP

$\Rightarrow$ CR = BP [As, AP = CR]

Now, BP = CR and BP $\parallel$ CR

$\Rightarrow$ BCRP is a parallelogram.

[When any pair of opposite sides are equal and parallel, the quadrilateral is a parallelogram]

BCRP is a parallelogram and opposite sides of a parallelogram are equal and parallel.

$\therefore$ PR = BC and PR $\parallel$ BC

Since, PQ = QR

$\Rightarrow$ PQ = $\frac{1}{2}$PR

= $\frac{1}{2}$BC [As, PR = BC]

Also, PQ $\parallel$ BC [As, PR $\parallel$ BC]

$\therefore$ PQ $\parallel$ BC and P + $\frac{1}{2}$BC **Hence Proved.**

ALTERNATIVE METHOD :

Construction : Draw CR parallel to BA intersecting PQ produced at point R.

Proof : In ΔAPQ and ΔCRQ

AQ = CQ [Given]

∠AQP = ∠RQC [Vertically opposite angles]

And ∠PAQ = ∠RCQ [Alternate angles, as AB ∥ CR]

ΔAPQ ≅ ΔCRQ [By ASA]

⇒ CR = AP and QR = PQ [By cpctc]

Since, CR = AP and AP = PB

⇒ CR = PB

Also, CR ∥ PB [By construction]

∴ PBCR is a parallelogram [As, opposite sides PB and CR are equal and parallel]

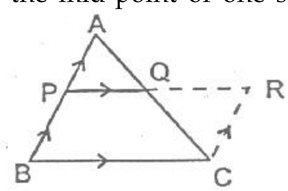
⇒ BC ∥ PR and BC = PR

⇒ BC ∥ PQ and BC = 2PQ [∵ PQ = QR]

⇒ PQ ∥ BC and PQ = $\frac{1}{2}$ BC **Hence Proved.**

(a) Converse of the Mid-Point Theorem

Statement : The line drawn through the mid-point of one side of a triangle parallel to the another side bisects the third side.



Given : A triangle ABC in which P is the mid-point of side AB nd PQ is parallel to BC.

To prove: PQ bisects the third side AB i.e., AQ = QC.

Construction : Through C, draw CR parallel to BA, which meets PQ produced at point R.

Proof : Since, PQ ∥ BC i.e., PR ∥ BC [Given] and CR ∥ BA i.e., CR ∥ BP [By construction]

∴ Opposite sides of quadrilateral PBCR are parallel.

⇒ PBCR is a parallelogram

⇒ BP = CR

Also, BP = AP [As, P is mid-point of AB]

∴ CR = AP

∴ AB ∥ CR and AC is transversal, ∠PAQ = ∠RCQ [Alternate angles]

∴ AB ∥ CR and PR is transversal, ∠APQ = ∠CRQ [Alternate angles]

In ΔAPQ and ΔCRQ

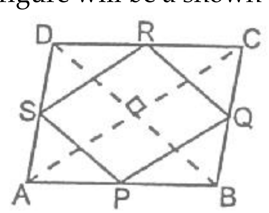
CR = AP, ∠PAQ = ∠RCQ and ∠APQ = ∠CRQ

⇒ ΔAPQ ≅ ΔCRQ [By ASA]

⇒ AAQ = QC **Hence Proved.**

Ex.1 ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Prove that the quadrilateral PQRS is a rectangle.

Sol. According to the given statement, the figure will be a shown alongside; using mid-point theorem :-



In ΔABC, $PQ \parallel AC$ and $PQ = \frac{1}{2} AC$ ....(i)

In ΔADC, $SR \parallel AC$ and $SR = \frac{1}{2} AC$ ....(ii)

$\therefore$ P = SR and $PQ \parallel SR$ [From (i) and (ii)]

$\Rightarrow$ PQRS is a parallelogram.

Now, PQRS will be a rectangle if any angle of the parallelogram PWRS is 90^0

$PQ \parallel AC$ [By mid-point theorem]

QR = BD [By mid-point theorem]

But, $AC \perp BD$ [Diagonals of a rhombus are perpendicular to each other]

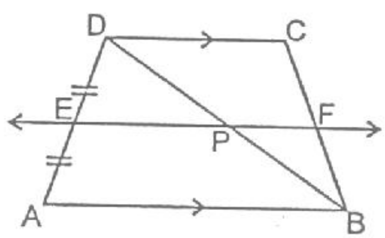
$\therefore$ $PQ \perp QR$ [Angle between two lines = angles between their parallels]

$\Rightarrow$ **PQRS is a rectangle** **Hence Proved.**

Ex.2 ABCD is a trapezium in which $AB \parallel DC$, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (as shown). Prove that F is the mid-point of BC.

Sol. Given line EF is parallel to AB and $AB \parallel DC$

$\therefore$ $EF \parallel AB \parallel DC$.



According to the converse of the mid-point theorem, is ΔABD, E is the mid-point of AD.

EP is parallel to AB [As $EF \parallel AB$]

$\therefore$ P is mid-point of side BD

[The line through the mid-point of a side of a triangle and parallel to the other side, bisects the third side]

Now, in ΔBCD, P is mid-point of BD [Proved above]

And, PF is parallel to DC [As $EF \parallel DC$]

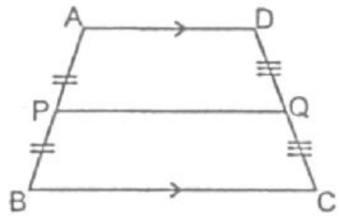
$\therefore$ F is mid-point of BC

[The line through the mid-point of a side of a triangle and parallel to the other side, bisects the third side]

Hence Proved.

REMARK :

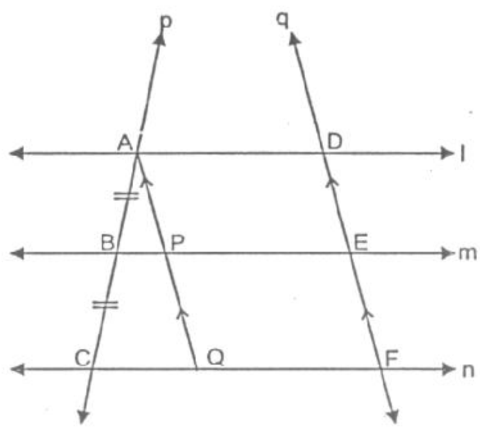
In quadrilateral ABCD, if side AD is parallel to side BC; ABCD is a trapezium.



Now, P and Q are the mid-points of the non-parallel sides of the trapezium; then PQ= $\frac{1}{2}$ (AD + BC). i.e. The length of the line segment joining the mid-points of the two non-parallel sides of a trapezium is always equal to half of the sum of the lengths of its two parallel sides.

Theorem.3: If there are three or more parallel lines and the intercepts made by them on a transversal are equal, then the corresponding intercepts on any other transversal are also equal.

Given : Three parallel lines I, m and n i.e., I ∥ m ∥ n. A transversal p meets these parallel lines are points A, B and C respectively such that AB = BC. Another transversal q also meets parallel lines I, m and n at points D, E and F respectively.



To Prove : DE = EF

Construction : Through point A, draw a line parallel to DEF; which meets BE at point P and CF and point Q.

Proof : In ΔACQ, B is mid-point of AC and BP is parallel to CQ and we know that the line through the mid-point of one side of the triangle and parallel to another sides bisects the third side.

∴ AP = PQ ...(i)

When the opposite sides of a quadrilateral are parallel, it is a parallelogram and so its opposite sides are equal.

∴ AP ∥ DE and AD ∥ PE ⇒ APED is a parallelogram.

⇒ AP = DE ....(ii)

And PQ ∥ EF and PE ∥ QF ⇒ PQFE is a parallelogram

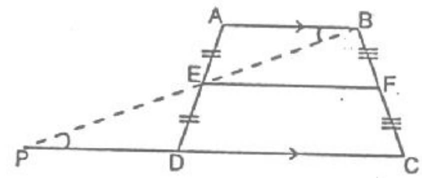
$\Rightarrow$ PQ = EF ....(iii)

From above equations, we get

DE = EF **Hence Proved.**

Ex.3 In the given figure, E and F are respectively, the mid-points of non-parallel sides of a trapezium ABCD.

Prove that



(i) EF ∥ AB

(ii) EF = $\frac{1}{2}$(AB + DC).

Sol. Join BE and produce it to intersect CD produced at point P. In ΔAEB and ΔDEP, AB ∥ PC and BP is transversal

$\Rightarrow$ ∠ABE = ∠DPE [Alternate interior angles]

∠AEB = ∠DEP [Vertically opposite angles]

And AE = DE [E is mid - point of AD]

$\Rightarrow$ ΔAEB ≅ ΔDEP [By ASA]

$\Rightarrow$ BE = PE [By cpctc]

And AB = DP [By cpctc]

Since, the line joining the mind-points of any two sides of a triangle is parallel and half of the third side, therefore, is ΔBPC,

E is mid-point of BP [As, BE = PE]

and F is mid-point of BC [Given]

$\Rightarrow$ EF ∥ PC and EF = $\frac{1}{2}$PC

$\Rightarrow$ EF ∥ DC and EF = $\frac{1}{2}$(PD + DC)

$\Rightarrow$ EF ∥ AB and EF = $\frac{1}{2}$(AB + DC) [As, DC ∥ AB and PD = AB] **Hence Proved.**