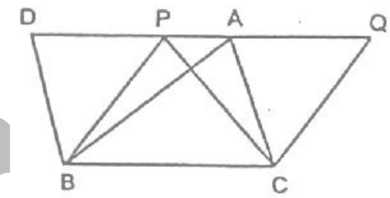
9.3 AREA OF A TRIANGLE

Theorem-5 : Two triangles on the same base (or equal bases) and between the same parallels are equal in area.

Given : Two triangles ABC and PCs on the same base BC and between the same parallel lines BC and AP.

To prove : ar(ΔABC) = ar(ΔPBC)

Construction : Through B, draw BD $\|$ CA intersecting PA produced in D and through C, draw CQ $\|$ BP, intersecting line AP in Q.



Proof : We have,

BD $\|$ CA [By construction]

And, BC $\|$ DA [Given]

$\therefore$ Quad. BCAD is a parallelogram.

Similarly, Quad. BCQP is a parallelogram.

Now, parallelogram BCQP and BCAD are on the same base BC, and between the same parallels.

$\therefore$ ar($\|^{gm}$ BCQP) = ar($\|^{gm}$ BCAD) ....(i)

We know that the diagonals of a parallelogram divides it into two triangles of equal area.

$\therefore$ ar(ΔPBC= $\frac{1}{2}$ ar($\|^{gm}$ BCQP) ....(ii)

And ar(ΔABC) = $\frac{1}{2}$ ar($\|^{gm}$ BCAD) ....(iii)

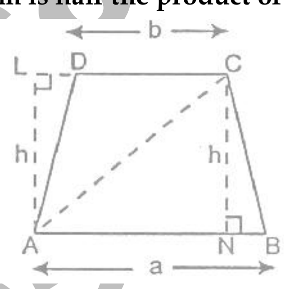
Now, ar($\|^{gm}$ BCQP) = ar($\|^{gm}$ BCAD) [From (i)]

$\Rightarrow$ $\frac{1}{2}$ ar($\|^{gm}$ BCAD) = $\frac{1}{2}$ ar($\|^{gm}$ BCQP)

Hence, ar(ΔABC) = ar(ΔPBC) [Using (ii) and (iii)]

Hence Proved.

Theorem-6 : The area of a trapezium is half the product of its height and the sum of the parallel sides.



Given : Trapezium ABCD in which AB $\|$ DC, AL $\perp$ DC, CN $\perp$ AB and AL = CN = h (say) AB = a, DC = b.

To prove : ar(trap. ABCD) = $\frac{1}{2}$h × (a + b).

Construction : Join AC.

Proof : AC is a diagonal of quad. ABCD.

$\therefore$ ar(trap. ABCD) = ar(ΔABC) + ar(ΔACD) = $\frac{1}{2}h \times a + \frac{1}{2}h \times b = \frac{1}{2}h(a+b)$. **Hence Proved.**

Theorem -7: Triangles having equal areas and having one side of the triangle equal to corresponding side of the other, have their corresponding altitudes equal/

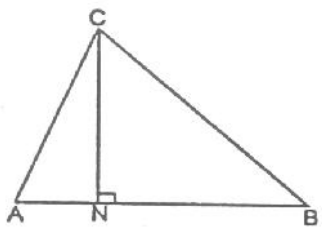
Given : Two triangles ABC and PQR such that (i) ar (ΔABC) = ar(ΔPQR) and (ii) AB = PQ.

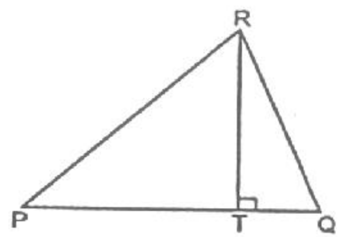
CN and RT and the altitude corresponding to AB and PQ respectively of the two triangles.

To prove : CR = RT.

Proof : In ΔABC, CN is the altitude corresponding to the side AB.

ar(ΔABC) = $\frac{1}{2}$ AB × CN ....(i)





Similarly, ar(ΔPQR) = $\frac{1}{2}$ PQ × RT ....(ii)

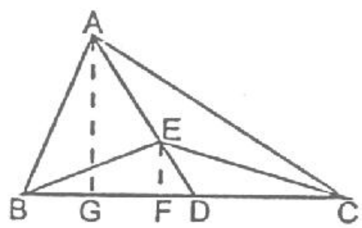
Since ar(ΔABC) = ar(ΔPQR) [Given]

$\therefore$ $\frac{1}{2}$ AB × CN = $\frac{1}{2}$ PQ × RT

Also, AB = PQ [Given]

CN = RT **Hence Proved.**

Ex.5 In figure, E is any point on median AD of a ΔABC. Show that ar(ABE) = ar(ACE).



Sol. **Construction :** From A draw AG $\perp$ BC and from E draw EF $\perp$ BC.

Proof : ar(ΔABD) = $\frac{BD \times AG}{2}$

ar(ΔADC) = $\frac{DC \times G}{2}$

But, BD = DC [$\therefore$ D is the mid-point of BC, AD being the median]

ar(ΔABD) = ar(ΔADC) ...(i)

Again, ar(ΔEBD)= $\frac{BD \times EF}{2}$

$$\text{ar}(\Delta EDC) = \frac{DC \times EF}{2}$$

But, BD = DC

$\therefore$ ar(ΔEBD) = ar(ΔEDC) ...(ii)

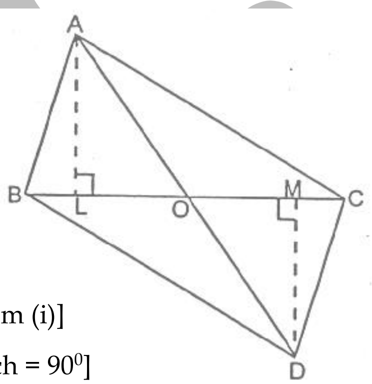
Subtracting (ii) from (i), we get

ar(ΔABD) - ar(ΔEBD) = ar(ΔADC) - ar(ΔEDC)

$\Rightarrow$ ar(ΔABE) = ar(ΔACE). **Hence Proved.**

Ex.6 Triangles ABC and DBC are on the same base BC; with A, D on opposite sides of the line BC, such that ar(ΔABC) = ar(ΔDBC). Show that BC bisects AD.

Sol. **Construction :** Draw AL $\perp$ BC and DM $\perp$ BC.



Proof : ar(ΔABC) = ar(ΔDBC) [Given]

$$\Rightarrow \frac{BC \times AL}{2} = \frac{BC \times DM}{2}$$

$\Rightarrow$ AL = DM ....(i)

Now in Δs OAL and OMD

AL = DM [From (i)]

$\Rightarrow$ $\angle$ALO = $\angle$DMO [Each = 90^0]

$\Rightarrow$ $\angle$AOL = $\angle$MOD [Vert. opp. $\angle$s]

$\Rightarrow$ $\angle$OAL = $\angle$ODM [Third angles of the triangles]

$\therefore$ ΔOAL $\cong$ ΔOMD [By ASA]

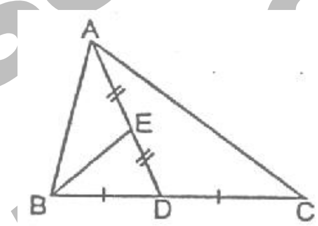
$\therefore$ OA = OD [By cpctc]

i.e., BC bisects AD. **Hence Proved.**

Ex.7 ABC is a triangle in which D is the mid-point of BC and E is the mid-point of AD. Prove that the area of ΔBED= $\frac{1}{4}$ area of ΔABC.

Sol. **Given : A aABC** in which D is the mid-point of BC and E is the mid-point of AD.

To prove: ar(ΔBED) = $\frac{1}{4}$ ar(ΔABC).



Proof : $\because$ AD is a median of ΔABC.

$\therefore$ ar(ΔABD) = ar(ΔADC) = $\frac{1}{2}$ ar(ΔABC) .....(i)

[$\therefore$ Median of a triangle divides it into two triangles of equal area) = $\frac{1}{2}$ ar(ΔABC)

Again,

$\because$ BE is a median of ΔABD,

$\therefore$ ar(ΔBEA) = ar(ΔBED) = $\frac{1}{2}$ ar(ΔABD)

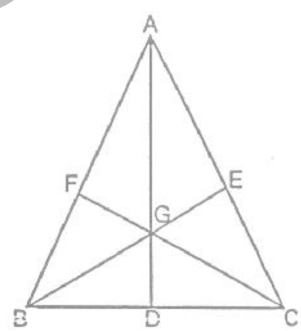
[$\therefore$ Median of a triangle divides it into two triangles of equal area]

And $\frac{1}{2}$ ar(ΔABD) = $\frac{1}{2} \times \frac{1}{2}$ ar(ΔABC) [From (i)]

$\therefore$ ar(ΔBED) = $\frac{1}{4}$ ar(ΔABC). **Hence Proved.**

Ex.8 if the medians of a ΔABC intersect at G, show that ar(ΔAGB) = ar(ΔBGC) = $\frac{1}{3}$ ar(ΔABC).

Sol. **Given :** A ΔABC its medians AD, BE and CF intersect at G.



To prove : ar(ΔAGB) = ar(ΔAGC) = ar(ΔBGC) = $\frac{1}{3}$ ar(ΔABC).

Proof : A median of triangle divides it into two triangles of equal area.

In ΔABC, AD is the median.

$\therefore$ ar(ΔABD) = ar(ΔACD) ...(i)

In ΔGBC, GD is the median.

$\therefore$ ar(ΔGBD) = ar(ΔGCD) ...(ii)

From (i) and (ii), we get

ar(ΔABD) - ar(ΔGBD) = ar(ΔACD) -ar(ΔGCD)

$\therefore$ a(ΔAGB) = ar(ΔAGC).

Similarly,

ar(ΔAGB) =ar(ΔAGC) =ar(ΔBGC) ....(iii)

But, ar(ABC) = ar(ΔAGB) + ar(ΔAGC) + ar(ΔBGC)

= 3 ar(ΔAGB) [Using (iii)]

$\therefore$ ar(ΔAGB) = $\frac{1}{3}$ ar(ΔABC).

Hence, ar(ΔAGB) = ar(ΔAGC) = arΔ(BGC) = $\frac{1}{3}$ ar(ΔABC). **Hence proved.**

Ex.9 **D,E** and F are respectively the mid points of the sides BC, CA and AB of a ΔABC. Show that

(i) BDEF is parallelogram

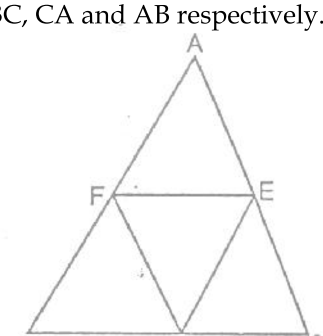
(ii) ar($\|^{gm}$ BDEF) = $\frac{1}{2}$ ar(ΔABC)

(iii) ar(ΔDEF) = $\frac{1}{4}$ ar(ΔABC)

Sol. **Given :** A ΔABC in which D,E,F are the mid-point of the side BC, CA and AB respectively.

To prove:

(i) Quadrilateral BDEF is parallelogram.



(ii) ar($\|^{gm}$ BDEF) = $\frac{1}{2}$ ar(ΔABC).

(iii) ar(ΔDEF) = $\frac{1}{4}$ ar(ΔABC).

Proof:

(i) In ΔABC,

∴ F is the mid-point of side AB and E is the mid point of side AC.

∴ EF ∥ BD

[∵ Line joining the mid-points of any two sides of a Δ is parallel to the third side.]

Similarly,

ED ∥ FB.

Hence, BDEF is a parallelogram. **Hence Proved.**

(ii) Similarly, we can prove that AFDE and FDCE are parallelograms.

∴ FD is diagonals of parallelogram BDEF.

∴ ar(ΔFBD) = ar(ΔDEF) ...(i)

Similarly,

ar(ΔFAE) = ar(ΔDEF) . ..(ii)

And ar(ΔDCE) = ar(ΔDEF) ...(iii)

From above equations, we have

ar(ΔFBD) = ar(ΔFAE) = ar(ΔDCE) = ar(ΔDEF)

And ar(ΔFBD) + ar(ΔDCE) + ar(ΔDEF) + ar(ΔFAE) = ar(ΔABC)

⇒ 2[ar(ΔFBD) + ar(ΔDEF)] = ar(ΔAC) [By using (i), (ii) and (iii)]

⇒ 2[ar($\|^{gm}$BDEF)] = ar(ΔABC)

⇒ ar($\|^{gm}$ BDEF) = $\frac{1}{2}$ ar(ABC)

(iii) Since, ΔABC is divided into four non-overlapping triangles FBD, FAE, DCE and DEF.

∴ ar(ΔABC) = ar(ΔFBD) + ar(ΔFAE) + ar(ΔDCE) + ar(ΔDEF)

⇒ ar(ΔABC) = 4 ar(ΔDEF) [Using (i), (ii) and (iii)]

⇒ ar(ΔDEF) = $\frac{1}{2}$ ar(ΔABC). **Hence Proved.**

Ex.10 Prove that the area of an equilateral triangle is equal to $\frac{\sqrt{3}}{4}a^2$, where a is the side of the triangle.

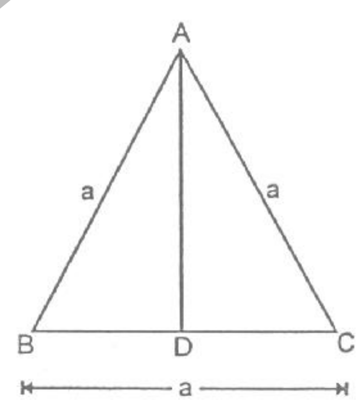
Sol. Draw AD ⊥ BC

⇒ ΔABD ≅ ΔACD [Br R.H.S.]

∴ BD = DC [By cpctc]

∴ BC = a

∴ BD = DC = $\frac{a}{2}$



In right angled ΔABD

$$AD^2 = AB^2 - BD^2 = a^2 - \left(\frac{a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\Rightarrow \quad AD = \frac{\sqrt{3}a}{2}$$

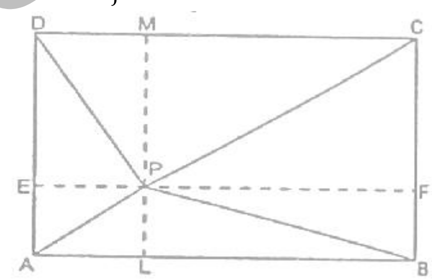
Area of $\Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a \frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4}$. **Hence Proved.**

Ex.11 In figure, P is a point in the interior of rectangle ABCD. Show that

(i) $ar(\Delta APB) + ar(\Delta PCD) = \frac{1}{2} ar(rect.\ ABCD)$

(ii) $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

Sol. **Given :** A rect. ABCD and P is a point inside it. PA, PB, PC and PD have been joined.



To prove :

(i) $ar(\Delta APB) + ar(\Delta PCD) = \frac{1}{2} ar(rect.\ ABCD)$

(ii) $ar(\Delta APD) + ar(\Delta BPC) = ar(\Delta APB) + ar(\Delta CPD)$.

Construction : Draw EPF $\parallel$ AB and LPM $\parallel$ AD.

Proof : (i) EPF $\parallel$ AB and DA cuts them,

$\therefore \quad \angle DEP = \angle EAB = 90^0$ [Corresponding angles]

$\therefore \quad PE \perp AD$.

Similarly, $PR \perp BC$; $PL \perp AB$ and $PM \perp DC$.

$\therefore \quad ar(\Delta APD) + ar(\Delta BPC)$

$$= \left(\frac{1}{2} \times AD \times PE\right) + ar\left(\frac{1}{2} \times BC \times PF\right) = \frac{1}{2} AD \times (PE + PF) \quad [\therefore BC = AD]$$

$$= \frac{1}{2} \times AD \times EF = \frac{1}{2} \times AD \times AB \quad [\therefore EF = AB]$$

$$= \frac{1}{2} \times (rectangle\ ABCD).$$

(ii) $ar(\Delta APB) + ar(PCD)$

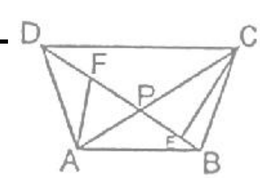
$$= \left(\frac{1}{2} \times AB \times PL\right) + \left(\frac{1}{2} \times DC \times PM\right) = \frac{1}{2} \times AB \times (PL + PM) \quad [\therefore EF = AB]$$

$$= \frac{1}{2} \times AB \times LM = \frac{1}{2} \times AB \times AD \quad [\because LM = AD]$$

$$= \frac{1}{2} \times ar(rect.\ ABCD).$$

$ar(\Delta APD) + ar(PBC) = ar(\Delta APB) + ar(PCD)$ **Hence Proved.**

Ex.12 Diagonals AC and BD of a quadrilateral ABCD intersect each other at P. Show that :



ar(APB) + ar(CPD) = ar(APD) × ar(BPC)

Sol. Draw perpendiculars AF and CE on BD.

$$ar(APB) \times ar(CPD) = \left(\frac{1}{2} \times PB \times AF\right) \times \left(\frac{1}{2} \times PD \times CE\right) \quad ....(i)$$

$$ar(APD) \times ar(BPC) = \left(\frac{1}{2} \times PD \times AF\right) \times \left(\frac{1}{2} \times BP \times CE\right) \quad ....(ii)$$

From above equations, we get

ar(APB) × ar(CPD) = ar(APD) × ar(BPC) **Hence Proved.**