11.4 Ellipse

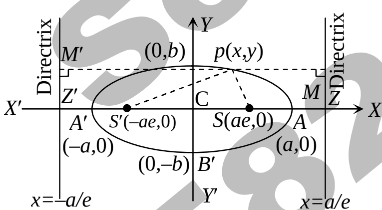
Definition

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point is in constant ratio (<1) to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity** of the ellipse, denoted by (e).

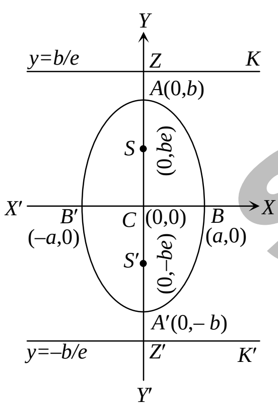
Standard equation of the ellipse

Let S be the focus, ZM be the directrix of the ellipse and $P(x,y)$ is any point on the ellipse, then by definition $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2(1-e^2)$.

Since $e<1$, therefore $a^2(1-e^2)<a^2 \Rightarrow b^2<a^2$.



The other form of equation of ellipse is $\frac{x^2}{y^2}+\frac{y^2}{b^2}=1$, where, $a^2=b^2(1-e^2) i.e., a<b$.



Difference between both ellipses will be clear from the following table:

Table : 18.11

Ellipse / **Imp. terms**	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For $a > b$	**For $b > a$**
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus-rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$ $(0 \le \theta < 2\pi)$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$2a/e$	$2b/e$
Tangents at the vertices	$x = -a$, $x = a$	$y = b$, $y = -b$

Parametric form of the ellipse

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the equation of ellipse in the parametric form will be given by $x = a\cos\theta, y = b\sin\theta$, where θ is the eccentric angle whose value vary from $0 \le \theta < 2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\cos\theta, b\sin\theta)$.

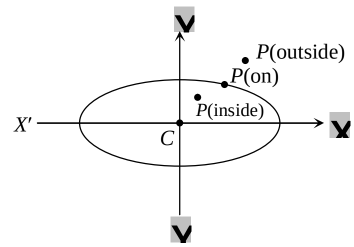
Special forms of an ellipse

(1) If the centre of the ellipse is at point (h,k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(2) If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $lx+my+n=0$ and $mx-ly+p=0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$, $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

Position of a point with respect to an ellipse

Let $P(x_1,y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse. The point lies outside, on or inside the ellipse as if $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >, =, < 0$



Intersection of a line and an ellipse

The line $y=mx+c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two distinct points if $a^2m^2+b^2>c^2$, in one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$.

Equations of tangent in different forms

(1) **Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1,y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(2) **Slope form:** If the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=a^2m^2+b^2$. Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to the ellipse.

Points of contact: Line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$.

(3) **Parametric form:** The equation of tangent at any point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.