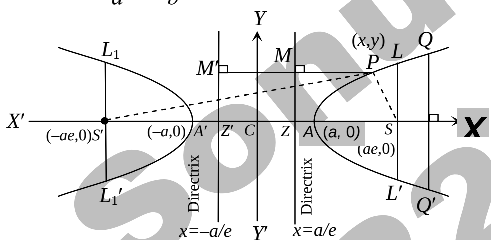
11.5 Hyperbola

Definition

A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.

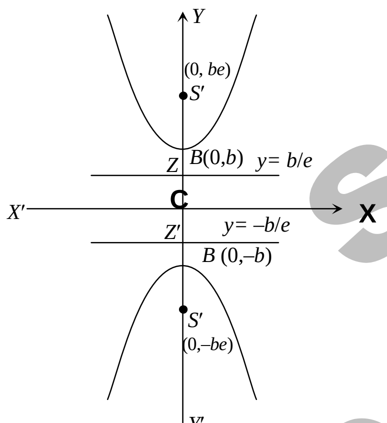
Standard equation of the hyperbola

Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2=a^2(e^2-1)$.



Conjugate hyperbola

The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of the given hyperbola.



Difference between both hyperbolas will be clear from the following table:

Table : 18.12

I. Hyperbola / Imp. terms	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm a/e$	$y=\pm b/e$
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a\sec\theta, b\tan\theta)$ $0\le\theta<2\pi$	$(b\sec\theta, a\tan\theta)$ $0\le\theta<2\pi$
Focal radii	$SP=ex_1-a$ $S'P=ex_1+a$	$SP=ey_1-b$ $S'P=ey_1+b$
Difference of focal radii $(S'P-SP)$	$2a$	$2b$
Tangents at the vertices	$x=-a, x=a$	$y=-b, y=b$
Equation of the transverse axis	$y=0$	$x=0$
Equation of the conjugate axis	$x=0$	$y=0$

Special form of hyperbola

If the centre of hyperbola is (h, k) and axes are parallel to the co-ordinate axes, then its equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

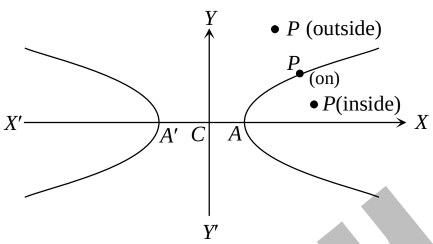
Parametric equations of hyperbola

The equations $x=a\sec\theta$ and $y=b\tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. This $(a\sec\theta, b\tan\theta)$ lies on the hyperbola for all values of θ.

Position of a point with respect to a hyperbola

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Then $P(x_1, y_1)$ will lie inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is positive, zero or negative.



Intersection of a line and a hyperbola

The straight line $y=mx+c$ will cut the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in two points may be real, coincident or imaginary according as $c^2 >,=,< a^2m^2-b^2$.

Condition of tangency : If straight line $y=mx+c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2=a^2m^2-b^2$.

Equations of tangent in different forms

(1) **Point form:** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(2) **Parametric form :** The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a \sec ¤, b \tan ¤)$ is $\frac{x}{a}\sec ¤ - \frac{y}{b}\tan ¤ = 1$.

(3) **Slope form :** The equations of tangents of slope *m* to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.