12.3 Direction cosines and direction ratios

(1) **Direction cosines:** If ~, , ¥ be the angles which a given directed line makes with the positive direction of the *x*, *y*, *z* co-ordinate axes respectively, then cos~, cos , cos¥ are called the direction cosines of the given line and are generally denoted by *l*, *m*, *n* respectively.

Thus, $l = \cos$~ , $m = \cos$ and $n = \cos$¥ , $l^2 + m^2 + n^2 = 1$.

By definition, it follows that the direction cosine of the axis of *x* are respectively $\cos 0^o$, $\cos 90^o$, $\cos 90^o$ *i.e.*, (1,0,0). Similarly direction cosines of the axes of *y* and *z* are respectively (0,1,0) and (0, 0, 1).

(2) **Direction ratios:** If *a*, *b*, *c* are three numbers proportional to direction cosines *l*, *m*, *n* of a line, then *a*, *b*, *c* are called its direction ratios. They are also called direction numbers or direction components.

Hence by definition,

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the sign should be taken all positive or all negative.

Direction ratios are not unique, whereas d.c.'s are unique. *i.e.*, $a^2 + b^2 + c^2 \neq 1$.

(3) **D.c.'s and d.r.'s of a line joining two points :** The direction ratios of line *PQ* joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$, $y_2 - y_1 = b$ and $z_2 - z_1 = c$, (say).

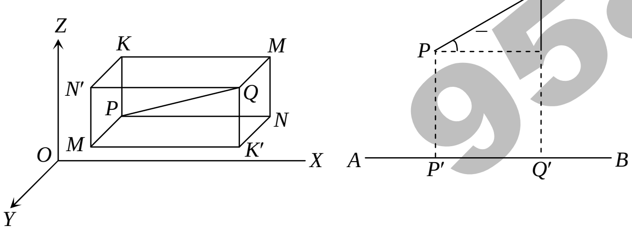
Then direction cosines are,

$$l = \frac{(x_2 - x_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, m = \frac{(y_2 - y_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}, n = \frac{(z_2 - z_1)}{\sqrt{\Sigma(x_2 - x_1)^2}}$$

i.e., $l = \frac{x_2 - x_1}{PQ}, m = \frac{y_2 - y_1}{PQ}, n = \frac{z_2 - z_1}{PQ}$.

Projection

Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are *l*, *m* and *n* : Let *PQ* be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$ and *AB* be a given line with d.c.'s as *l*, *m*, *n*. If the line segment *PQ* makes angle ¯ with the line *AB*, then



Projection of PQ is $P'Q' = PQ\cos\theta$

$= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$

$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

For x-axis, $l = 1$, $m = 0$, $n = 0$.

Hence, projection of PQ on x-axis $= x_2 - x_1$.

Similarly, projection of PQ on y-axis $= y_2 - y_1$ and projection of PQ on z-axis $= z_2 - z_1$.

Angle between two lines

Let θ be the angle between two straight lines AB and AC whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

If direction ratios of two lines a_1, b_1, c_1 and a_2, b_2, c_2 are given, then angle between two lines is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}} .$$

Particular results: We have, $\sin^2\theta = 1 - \cos^2\theta$

$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$

$= (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$

$\Rightarrow \sin\theta = \pm\sqrt{\Sigma(l_1 m_2 - l_2 m_1)^2}$, which is known as Lagrange's identity.

The value of $\sin\theta$ can easily be obtained by,

$$\sin\theta = \sqrt{\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2 + \begin{vmatrix} m_1 & n_1 \\ n_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2}$$

If a_1, b_1, c_1 and a_2, b_2, c_2 are d.r.'s of two given lines, then angle θ between them is given by $\sin\theta = \dfrac{\sqrt{\Sigma(a_1 b_2 - a_2 b_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Condition of perpendicularity: If the given lines are perpendicular, then $\theta = 90^\circ$ *i.e.*, $\cos\theta = 0$

$\Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Condition of parallelism: If the given lines are parallel, then $\theta = 0^o$ *i.e.*, $\sin\theta = 0 \Rightarrow \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

Similarly, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.