CHAPTER – 13
Limits, Continuity and Differentiability

13.1 Introduction

Limit of a function

Let $y = f(x)$ be a function of *x*. If at $x = a, f(x)$ takes indeterminate form, then we consider the values of the function which are very near to '*a*'. If these values tend to a definite unique number as *x* tends to '*a*', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

(1) Left hand and right hand limit: Consider the values of the functions at the points which are very near to *a* on the left of *a*. If these values tend to a definite unique number as *x* tends to *a*, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$.

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$.

(2) Method for finding L.H.L. and R.H.L.

(i) For finding right hand limit (R.H.L.) of the function, we write *x* + *h* in place of *x*, while for left hand limit (L.H.L.) we write *x* – *h* in place of *x*.

(ii) Then we replace *x* by '*a*' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(3) Existence of limit: $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Fundamental theorems on limits

The following theorems are very useful for evaluation of limits if $\lim_{x \to 0} f(x) = l$ and $\lim_{x \to 0} g(x) = m$ (*l* and *m* are real numbers) then

(1) $\lim_{x \to a} (f(x) + g(x)) = l + m$ (Sum rule)

(2) $\lim_{x \to a} (f(x) - g(x)) = l - m$ (Difference rule)

(3) $\lim_{x\to a}(f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x\to a} k\ f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a}\dfrac{1}{f(x)} = 0$

(7) $\lim_{x\to a}\log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$

(8) If $f(x) \le g(x)$ for all x, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(9) $\lim_{x\to a}[f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(10) If p and q are integers, then $\lim_{x\to a}(f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. $e.g.$ $\lim_{x\to a}\ln[f(x)] = \ln(l)$, only if $l > 0$.

Methods of evaluation of limits

We shall divide the problems of evaluation of limits in five categories.

(1) Algebraic limits: Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x\to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method:** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method:** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method:** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when *x* ì ã:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

(2) Trigonometric limits: To evaluate trigonometric limit the following results are very important.

(i) $\lim_{x\to 0}\dfrac{\sin x}{x} = 1 = \lim_{x\to 0}\dfrac{x}{\sin x}$

(ii) $\lim_{x\to 0}\dfrac{\tan x}{x} = 1 = \lim_{x\to 0}\dfrac{x}{\tan x}$

(iii) $\lim_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0}\dfrac{x}{\sin^{-1} x}$

(iv) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$ (vi) $\lim_{x \to 0} \cos x = 1$

(vii) $\lim_{x \to a} \frac{\sin(x-a)}{x-a} = 1$ (viii) $\lim_{x \to a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a, |a| \le 1$

(x) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a; |a| \le 1$

(xi) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a; -\infty < a < \infty$

(xii) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x \to \infty} \frac{\sin(1/x)}{(1/x)} = 1$

(3) Logarithmic limits: To evaluate the logarithmic limits we use following formulae

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots \text{to } \infty$ where $-1 < x \le 1$ and expansion is true only if base is e.

(ii) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(iii) $\lim_{x \to e} \log_e x = 1$

(iv) $\lim_{x \to 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \ne 1$

(4) Exponential limits

(i) Based on series expansion

We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots \ldots\infty$

To evaluate the exponential limits we use the following results

(a) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ (b) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(c) $\lim_{x \to 0} \frac{e^{\alpha x} - 1}{x} = \alpha \quad (\alpha \ne 0)$

(ii) Based on the form 1^∞ : To evaluate the exponential form 1^∞ we use the following results.

(a) If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$\lim_{x \to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$ or when $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$.

Then $\lim_{x \to a} \{f(x)\}^{g(x)} = \lim_{x \to a} [1 + f(x) - 1]^{g(x)} = e^{\lim_{x \to a} (f(x)-1) g(x)}$

(b) $\lim_{x \to 0} (1+x)^{1/x} = e$ (c) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(d) $\lim_{x \to 0} (1 + \alpha x)^{1/x} = e^\alpha$ (e) $\lim_{x \to \infty} \left(1 + \frac{\alpha}{x}\right)^x = e^\alpha$

- $\lim_{x\to\infty} a^x = \begin{cases} \infty, \text{if } a > 1 \\ 0, \text{if } a < 1 \end{cases}$ *i.e.*, $a^\infty = \infty$, if $a > 1$ and $a^\infty = 0$ if $a < 1$.

(5) L-Hospital's rule : If $f(x)$ and $g(x)$ be two functions of *x* such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$.

The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.