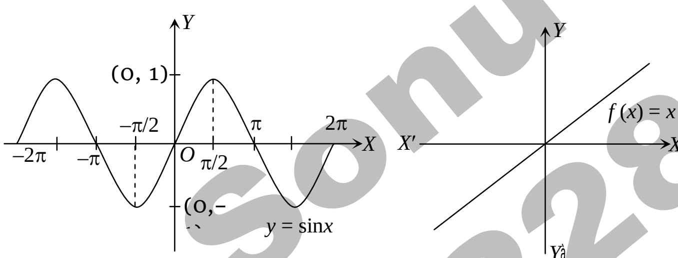
13.2 Continuity

Introduction

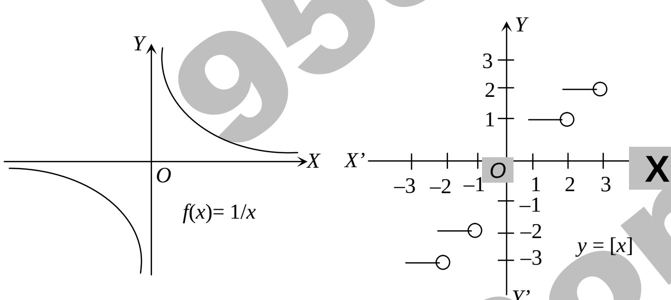
The word 'continuous' means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be continuous.

A function which is not continuous is called a discontinuous function. While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function $[x]$ has break at every integral point, so it is not continuous. Similarly $\tan x,\ \cot x,\ \sec x$, $\frac{1}{x}$ etc. are also discontinuous function.

Continuous function



Discontinuous function



Continuity of a function at a point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('a' lies in the domain of f)

(2) $\lim_{x \to a} f(x)$ exist *i.e.* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x \to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity: A function f is said to be continuous at a point *a* of its domain *D* if for every $\varepsilon > 0$ there exists $\delta > 0$ (dependent on ε) such that $|x-a| < \delta$ $\Rightarrow |f(x) - f(a)| < \varepsilon$.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$ *i.e.*, if $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a+} f(x)$.

Continuity from left and right

Function $f(x)$ is said to be

(1) Left continuous at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

Properties of continuous functions: Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line *R i.e.* $(-\infty, \infty)$. *e.g.*, polynomial function, e^x, $\sin x, \cos x$, constant, x^n, $|x-a|$ etc.

(ii) Integral function of a continuous function is a continuous function.

(iii) If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$ then $(fog)(x)$ is continuous at $x = a$.

(iv) If $f(x)$ is continuous in a closed interval $[a,b]$ then it is bounded on this interval.

(v) If $f(x)$ is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there is atleast one value of x for which $f(x)$ vanishes. *i.e.* if $f(a) > 0$, $f(b) < 0 \Rightarrow \exists c \in (a, b)$ such that $f(c) = 0$.

Discontinuous function

(1) **Discontinuous function :** A function '*f*' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point '*a*' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.