3.4 Formulae for the trigonometric ratios of sum and differences of two angles

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(2) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(3) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(4) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(5) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(6) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(7) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

(8) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(9) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(10) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(11) $\tan A \pm \tan B = \dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B} = \dfrac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B}$

$$= \frac{\sin(A \pm B)}{\cos A . \cos B}, \left(A \neq n\text{®} + \frac{\text{®}}{2}, B \neq m\text{®} \right)$$

(12) $\cot A \pm \cot B = \dfrac{\sin(B \pm A)}{\sin A \sin B}, \left(A \neq n\text{®}, B \neq m\text{®} + \dfrac{\text{®}}{2} \right)$

Formulae for the trigonometric ratios of sum and differences of three angles

(1) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

or sin $(A+B+C) = \cos A \cos B \cos C(\tan A + \tan B + \tan C - \tan A . \tan B . \tan C)$

(2) $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

$\cos(A+B+C) = \cos A \cos B \cos C(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

(3) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(4) $\cot(A+B+C) = \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C . \cot A - 1}$

In general :

(5) $\sin(A_1 + A_2 + \ldots\ldots + A_n) = \cos A_1 \cos A_2 \ldots\ldots \cos A_n (S_1 - S_3 + S_5 - S_7 + \ldots)$

(6) $\cos(A_1 + A_2 + .. + A_n) = \cos A_1 \cos A_2 .. \cos A_n (1 - S_2 + S_4 - S_6 ..)$

(7) $\tan(A_1 + A_2 + \ldots.. + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots.}{1 - S_2 + S_4 - S_6 + \ldots.}$

where, $S_1 = \tan A_1 + \tan A_2 + \ldots. + \tan A_n$ = The sum of the tangents of the separate angles.

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots.$ = The sum of the tangents taken two at a time.

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \dots$ = Sum of tangents three at a time, and so on.

If $A_1 = A_2 = \dots = A_n = A$, then $S_1 = n \tan A$,

$S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A, \dots$

(8) $\sin nA = \cos^n A({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \dots)$

(9) $\cos nA = \cos^n A(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - \dots)$

(10) $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \dots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + \dots}$

(11) $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A - \dots)$

(12) $\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A$
$+ {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A + \dots)$

(13) $\sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$
$= \dfrac{\sin\{\alpha + (n-1)(\beta/2)\} . \sin(n\beta/2)}{\sin(\beta/2)}$

(14) $\cos(\alpha) + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta)$
$= \dfrac{\cos\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\} . \sin\left\{n\left(\frac{\beta}{2}\right)\right\}}{\sin\left(\frac{\beta}{2}\right)}$

Formulae to transform the product into sum or difference

(1) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(2) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

(3) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

(4) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

Let $A + B = C$ and $A - B = D$

Then, $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Therefore, we find out the formulae to transform the sum or difference into product.

(1) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(2) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

(3) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(4) $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2} = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$.

Trigonometric ratio of multiple of an angle

(1) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(2) $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$= \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$; where $A \neq (2n+1)\dfrac{®}{4}$.

(3) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(4) $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60^o - A).\sin A.\sin(60^o + A)$

(5) $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60^o - A).\cos A.\cos(60^o + A)$

(6) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60^o - A).\tan A.\tan(60^o + A)$,

where $A \neq n® + ® / 6$

(7) $\sin 4¯ = 4\sin ¯ .\cos^3 ¯ - 4\cos ¯ \sin^3 ¯$

(8) $\cos 4¯ = 8\cos^4 ¯ - 8\cos^2 ¯ + 1$

(9) $\tan 4¯ = \dfrac{4\tan ¯ - 4\tan^3 ¯}{1 - 6\tan^2 ¯ + \tan^4 ¯}$

(10) $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$

(11) $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

Trigonometric ratio of sub-multiple of an angle

(1) $\left|\sin\dfrac{A}{2} + \cos\dfrac{A}{2}\right| = \sqrt{1+\sin A}$

or $\sin\dfrac{A}{2} + \cos\dfrac{A}{2} = \pm\sqrt{1+\sin A}$

i.e., $\begin{cases} +, \text{If } 2n® - ® / 4 \leq A/2 \leq 2n® + \dfrac{3®}{4} \\ -, \text{otherwise} \end{cases}$

(2) $\left|\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right| = \sqrt{1-\sin A}$

or $(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}) = \pm\sqrt{1-\sin A}$

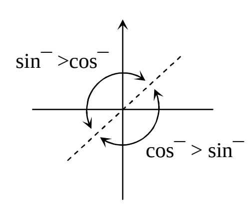
i.e., $\begin{cases} +, \text{If } 2n® + ® / 4 \leq A/2 \leq 2n® + \dfrac{5®}{4} \\ -, \text{otherwise} \end{cases}$

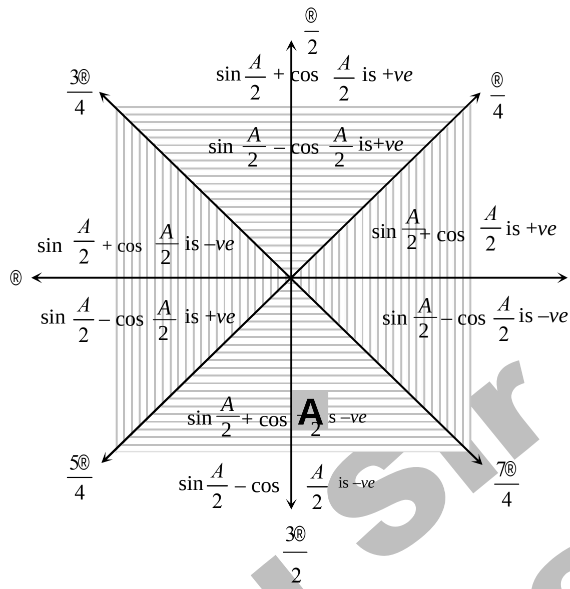
(3) (i) $\tan\dfrac{A}{2} = \dfrac{\pm\sqrt{\tan^2 A + 1} - 1}{\tan A} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A}$,

where $A \neq (2n+1)®$

(ii) $\cot\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{1-\cos A}} = \dfrac{1+\cos A}{\sin A}$, where $A \neq 2n®$

The ambiguities of signs are removed by locating the quadrants in which $\frac{A}{2}$ lies or you can follow the following figure,





(4) $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$; where $A \neq (2n+1)$®

(5) $\cot^2\frac{A}{2}=\frac{1+\cos A}{1-\cos A}$; where $A \neq 2n$®

Maximum and minimum value of *a* cos¯ + *b* sin¯

Let $a = r\cos$~ .....(i) and $b = r\sin$~ .....(ii)

Squaring and adding (i) and (ii), then $a^2+b^2=r^2$ or, $r=\sqrt{a^2+b^2}$

∴ $a\sin$¯ $+ b\cos$¯ $= r(\sin$¯ $\cos$~ $+ \cos$¯ $\sin$~$) = r\sin($¯ $+$~$)$

But $-1 \le \sin$¯ < 1 So, $-1 \le \sin($¯ $+$~$) \le 1$;

Then $-r \le r\sin($¯ $+$~$) \le r$

Hence, $-\sqrt{a^2+b^2} \le a\sin$¯ $+ b\cos$¯ $\le \sqrt{a^2+b^2}$

Then the greatest and least values of $a\sin$¯ $+ b\cos$¯ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

.

Therefore, $\sin^2 x + \operatorname{cosec}^2 x \ge 2$, for every real *x*.

$\cos^2 x + \sec^2 x \ge 2$, for every real *x*.

$\tan^2 x + \cot^2 x \ge 2$, for every real *x*.

Conditional trigonometrical identities

We have certain trigonometric identities.

Like, $\sin^2$¯ $+ \cos^2$¯ $= 1$ and $1 + \tan^2$¯ $= \sec^2$¯ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

(1) If $A+B+C=180^o$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(ii) $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$

(iii) $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4 \sin A \sin B \sin C$

(iv) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(v) $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$

(2) If $A + B + C = 180^o$, then

(i) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

(ii) $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$

(iii) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(iv) $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

(v) $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$

(3) If $A + B + C =$ ®, then

(i) $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$

(ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \cos C$

(4) If $A + B + C =$ ®, then

(i) $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(ii) $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

(iii) $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

(iv) $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

(5) If $x + y + z = \frac{®}{2}$, then

(i) $\sin^2 x + \sin^2 y + \sin^2 z = 1 - 2 \sin x \sin y \sin z$

(ii) $\cos^2 x + \cos^2 y + \cos^2 z = 2 + 2 \sin x \sin y \sin z$

(iii) $\sin 2x + \sin 2y + \sin 2z = 4 \cos x \cos y \cos z$

(6) If $A + B + C =$ ®, then

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

(iii) $\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1$

(iv) $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

FEW IMPORTANT POINTS

- The angle between two consecutive digits in a clock is 30^o (= ®/6 radians). The hour hand rotates through an angle of 30^o in one hour.
- The minute hand rotate through an angle of 6^o in one minute.
- $\sin n® = 0, \quad \cos n® = (-1)^n$
- $\sin(n® + ¯) = (-1)^n \sin ¯$, $\cos(n® + ¯) = (-1)^n \cos ¯$
- $\sin\left(\frac{n®}{2} + ¯\right) = (-1)^{\frac{n-1}{2}} \cos ¯$, if n is odd

  $= (-1)^{n/2} \sin ¯$, if n is even
- $\cos\left(\frac{n®}{2} + ¯\right) = (-1)^{\frac{n+1}{2}} \sin ¯, \quad \text{if } n \text{ is odd}$

  $= (-1)^{n/2} \cos ¯, \quad \text{if } n \text{ is even}$
- If $x = \sec ¯ + \tan ¯$, then

$\frac{1}{x} = \sec ¯ - \tan ¯$.

- If $x = \text{coesc}\, ¯ + \cot ¯$, then

$\frac{1}{x} = \text{cosec}\, ¯ - \cot ¯$.

- $\sin(60^o - ¯).\sin ¯ \sin(60^o + ¯) = \frac{1}{4}\sin 3¯$.
- $\cos(60 - ¯).\cos ¯ .\cos(60^o + ¯) = \frac{1}{4}\cos 3¯$.
- $\tan(60° - ¯).\tan ¯ .\tan(60° + ¯) = \tan 3¯$
- $\cos A.\cos 2A.\cos 2^2.A \cos 2^3 A....... \cos 2^{n-1} A$

  $= \frac{\sin 2^n A}{2^n \sin A}, \text{if } A \neq n®$

  $= 1$, if $A = 2n®$

  $= -1$, if $A = (2n+1)®$
- Any formula that gives the value of $\sin\frac{A}{2}$ in terms of sin A shall also give the value of sine of $\frac{n® + (-1)^n A}{2}$.
- Any formula that gives the value of $\cos\frac{A}{2}$ in terms of cos A shall also give the value of

cos of $\frac{2n\text{®} \pm A}{2}$.

✍ Any formula that gives the value of $\tan\frac{A}{2}$ in terms of tan A shall also give the value of tan of $\frac{n\text{®} \pm A}{2}$.

✍ $\cos\text{~} + \cos \quad + \cos\text{¥} + \cos(\text{~} + \quad + \text{¥})$

$= 4\cos\frac{\text{~} +}{2}\cos\frac{+\text{¥}}{2}\cos\frac{\text{¥} + \text{~}}{2}$

✍ $\sin\text{~} + \sin \quad + \sin\text{¥} - \sin(\text{~} + \quad + \text{¥})$

$= 4\sin\frac{\text{~} +}{2}\sin\frac{+\text{¥}}{2}\sin\frac{\text{¥} + \text{~}}{2}$

✍ $\tan\text{~} + 2\tan 2\text{~} + 4\tan 4\text{~} + 8\cot 8\text{~} = \cot\text{~}$

✍ Method of componendo and dividendo

If $\frac{p}{q} = \frac{a}{b}$, then by componendo and dividendo

We can write $\frac{p+q}{p-q} = \frac{a+b}{a-b}$ or $\frac{q+p}{q-p} = \frac{b+a}{b-a}$

or $\frac{p-q}{p+q} = \frac{a-b}{a+b}$ or $\frac{q-p}{q+p} = \frac{b-a}{b+q}$.