7.4 Combinations

Definition

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

Notation: The number of all combinations of n things, taken r at a time is denoted by $C(n,r)$ or ${}^{n}C_{r}$ or $\binom{n}{r}$.

${}^{n}C_{r}$ is always a natural number.

Difference between a permutation and combination :

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) Each combination corresponds to many permutations. For example, the six permutations *ABC, ACB, BCA, BAC, CBA* and *CAB* correspond to the same combination *ABC*.

Number of combinations without repetition

The number of combinations (selections or groups) that can be formed from n different objects taken $r(0 \le r \le n)$ at a time is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$. Also ${}^{n}C_{r} = {}^{n}C_{n-r}$.

Let the total number of selections (or groups) = x. Each group contains r objects, which can be arranged in r ! ways. Hence the number of arrangements of r objects = $x \times (r!)$. But the number of arrangements = ${}^{n}P_{r}$.

$$\Rightarrow x(r!) = {}^{n}P_{r} \Rightarrow x = \frac{{}^{n}P_{r}}{r!} \Rightarrow x = \frac{n!}{r!(n-r)!} = {}^{n}C_{r}.$$

Number of combinations with repetition and all possible selections

(1) The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times.

= Coefficient of x^{r} in $(1 + x + x^{2} + ... + x^{r})^{n}$

= Coefficient of x^{r} in $(1-x)^{-n} = {}^{n+r-1}C_{r}$

(2) The total number of ways in which it is possible to form groups by taking some or all of n things at a time is ${}^{n}C_{1} + {}^{n}C_{2} + ... + {}^{n}C_{n} = 2^{n} - 1$.

(3) The total number of ways in which it is possible to make groups by taking some or all out of $n = (n_1 + n_2 + ...)$ things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1 + 1)(n_2 + 1).....\} - 1$.

(4) The number of selections of *r* objects out of *n* identical objects is 1.

(5) Total number of selections of zero or more objects from *n* identical objects is $n+1$.

(6) The number of selections taking at least one out of $a_1 + a_2 + a_3 + ... + a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on.....a_n are alike (of n^{th} kind) and *k* are distinct

$= [(a_1 + 1)(a_2 + 1)(a_3 + 1).....(a_n + 1)]2^k - 1$.