8.6 Binomial theorem for any Index

Statement :

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{n(n-1)......(n-r+1)}{r!}x^r + ...\text{terms up to } \infty$$

when n is a negative integer or a fraction, where $-1 < x < 1$, otherwise expansion will not be possible.

If first term is not 1, then make first term unity in the following way, $(x+y)^n = x^n\left[1+\frac{y}{x}\right]^n$, if $\left|\frac{y}{x}\right| < 1$.

General term : $T_{r+1} = \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r$

Some important expansions

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)......(n-r+1)}{r!}x^r + ...$

(ii) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - .... + \frac{n(n-1)(n-2).....(n-r+1)}{r!}(-x)^r + ...$

(iii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}x^r + ...$

(iv) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ... + \frac{n(n+1)......(n+r-1)}{r!}(-x)^r + ...$

(v) $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...\infty$

(vi) $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...\infty$

(vii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...\infty$

(viii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...\infty$

(ix) $(1+x)^{-3} = 1 - 3x + 6x^2 - ....\infty$

(x) $(1-x)^{-3} = 1 + 3x + 6x^2 + ...\infty$

Problems on approximation by the binomial theorem : We have $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$.

If x is small compared with 1, we find that the values of $x^2, x^3, x^4, ...$ become smaller and smaller.

$\therefore$ The terms in the above expansion become smaller and smaller. If x is very small compared with 1, we might take 1 as a first approximation to the value of $(1+x)^n$ or $1+nx$ as a second approximation.

Three / Four consecutive terms or Coefficients

(1) **If consecutive coefficients are given:** In this case divide consecutive coefficients pair wise. We get equations and then solve them.

(2) **If consecutive terms are given :** In this case divide consecutive terms pair wise *i.e.* if four consecutive terms be $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ then find $\frac{T_r}{T_{r+1}}, \frac{T_{r+1}}{T_{r+2}}, \frac{T_{r+2}}{T_{r+3}} \Rightarrow a_1, a_2, a_3$ (say) then divide a_1 by a_2 and a_2 by a_3 and solve.

Some important points

(1) Pascal's Triangle

1	$(x+y)^0$
1 1	$(x+y)^1$
1 2 1	$(x+y)^2$
1 3 3 1	$(x+y)^3$
1 4 6 4 1	$(x+y)^4$
1 5 10 10 5 1	$(x+y)^5$

Pascal's triangle gives the direct binomial coefficients.

Example : $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.