9.4 Harmonic Progression (H.P.)

Definition

A progression is called a harmonic progression (H.P.) if the reciprocals of its terms are in A.P.

Standard form : $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots..$

Example: The sequence $1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{9},\ldots$ is a H.P., because the sequence 1, 3, 5, 7, 9, ….. is an A.P.

General term of an H.P.

If the H.P. be as $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},\ldots$ then corresponding A.P. is $a, a+d, a+2d, \ldots..$

T_n of A.P. is $a+(n-1)d$

$\therefore T_n$ of H.P. is $\frac{1}{a+(n-1)d}$

In order to solve the question on H.P., we should form the corresponding A.P. Thus, General term :

$T_n=\frac{1}{a+(n-1)d}$ or $T_n \text{ of H.P.}=\frac{1}{T_n \text{ of A.P.}}$.

Harmonic mean

If three or more numbers are in H.P., then the numbers lying between the first and last are called harmonic means (H.M.'s) between them. For example 1, 1/3, 1/5, 1/7, 1/9 are in H.P. So 1/3, 1/5 and 1/7 are three H.M.'s between 1 and 1/9.

Also, if *a*, *H*, *b* are in H.P., then *H* is called harmonic mean between *a* and *b*.

(1) **Insertion of harmonic means**

(i) Single H.M. between *a* and *b* $=\frac{2ab}{a+b}$.

(ii) *H*, H.M. of *n* non-zero numbers $a_1, a_2, a_3, \ldots, a_n$ is given by $\frac{1}{H}=\frac{\frac{1}{a_1}+\frac{1}{a_2}+\ldots..+\frac{1}{a_n}}{n}$.

(iii) Let *a*, *b* be two given numbers. If *n* numbers $H_1, H_2, \ldots\ldots H_n$ are inserted between *a* and *b* such that the sequence $a, H_1, H_2, H_3, \ldots\ldots H_n, b$ is a H.P., then $H_1, H_2, \ldots\ldots H_n$ are called *n* harmonic means between *a* and *b*.

Now, $a, H_1, H_2, H_3, \ldots\ldots H_n, b$ are in H.P.

$\Rightarrow \frac{1}{a},\frac{1}{H_1},\frac{1}{H_2},\ldots\ldots\frac{1}{H_n},\frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$\frac{1}{b} = (n+2)^{th}$ term $= T_{n+2}$

$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow D = \frac{a-b}{(n+1)ab}$.

Thus, if n harmonic means are inserted between two given numbers a and b, then the common difference of the corresponding A.P. is given by $D = \frac{a-b}{(n+1)ab}$.

Also, $\frac{1}{H_1} = \frac{1}{a} + D$, $\frac{1}{H_2} = \frac{1}{a} + 2D$,......., $\frac{1}{H_n} = \frac{1}{a} + nD$,

where $D = \frac{a-b}{(n+1)ab}$.

Properties of H.P.

(1) No term of H.P. can be zero.

(2) If H is the H.M. between a and b, then

(i) $\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$

(ii) $(H-2a)(H-2b) = H^2$

(iii) $\frac{H+a}{H-a} + \frac{H+b}{H-b} = 2$

Arithmetico-geometricProgression(A.G.P.)

Definition

The combination of arithmetic and geometric progression is called arithmetico-geometric progression.

nth term of A.G.P.

If $a_1, a_2, a_3, ..., a_n, ...$ is an A.P. and $b_1, b_2, ..., b_n, ...$ is a G.P., then the sequence $a_1b_1, a_2b_2, a_3b_3, ..., a_nb_n, ...$ is said to be an arithmetico-geometric sequence.

Thus, the general form of an arithmetico geometric sequence is $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, ...$

From the symmetry we obtain that the n^{th} term of this sequence is $[a+(n-1)d]r^{n-1}$.

Also, let $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, ...$ be an arithmetico-geometric sequence.

Then, $a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + ...$ is an arithmetico-geometric series.

Sum of A.G.P.

(1) **Sum of n terms :** The sum of n terms of an arithmetico-geometric sequence $a,(a+d)r,(a+2d)r^2,\ (a+3d)r^3,.....$ is given by $S_n=\begin{cases}\dfrac{a}{1-r}+dr\dfrac{(1-r^{n-1})}{(1-r)^2}-\dfrac{\{a+(n-1)d\}r^n}{1-r}, \text{when } r\neq 1\\ \dfrac{n}{2}[2a+(n-1)d], \text{when } r=1\end{cases}$

(2) **Sum of infinite sequence:** Let $|r|<1$. Then $r^n, r^{n-1}\to 0$ as $n\to\infty$ and it can also be shown that $n.r^n\to 0$ as $n\to\infty$. So, we obtain that $S_n\to\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}$, as $n\to\infty$.

In other words, when $|r|<1$ the sum to infinity of an arithmetico-geometric series is $S_\infty=\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}$.

Method for finding sum

This method is applicable for both sum of n terms and sum of infinite number of terms.

First suppose that sum of the series is S, then multiply it by common ratio of the G.P. and subtract. In this way, we shall get a G.P., whose sum can be easily obtained.

Method of difference

If the differences of the successive terms of a series are in A.P. or G.P., we can find n^{th} term of the series by the following steps :

Step I: Denote the n^{th} term by T_n and the sum of the series upto n terms by S_n.

Step II: Rewrite the given series with each term shifted by one place to the right.

Step III: By subtracting the later series from the former, find T_n.

Step IV: From T_n, S_n can be found by appropriate summation.

Example **:** Consider the series 1+ 3 + 6 + 10 + 15 +…..to n terms. Here differences between the successive terms are $3-1$, 6 – 3, 10 – 6, 15 – 10, …….*i.e.*, 2, 3, 4, 5,…… which are in A.P. This difference could be in G.P. also. Now let us find its sum

$$S=1+3+6+10+15+.....+T_{n-1}+T_n$$

$$S=\quad 1+3+6+10+.........+T_{n-1}+T_n$$

Subtracting, we get

$$0=1+2+3+4+5+.........+(T_n-T_{n-1})-T_n$$

$\Rightarrow \qquad T_n=1+2+3+4+.........$ to n terms.

$$\Rightarrow \qquad T_n=\frac{1}{2}n(n+1) \qquad \therefore\ S_n=\Sigma T_n=\frac{1}{2}[\Sigma n^2+\Sigma n]$$

$$=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]=\frac{n(n+1)(n+2)}{6}.$$