9.5 Miscellaneous series

Special series

(1) Sum of first *n* natural numbers

$$= 1 + 2 + 3 + \ldots\ldots + n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

(2) Sum of squares of first *n* natural numbers

$$= 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.$$

(3) Sum of cubes of first *n* natural numbers

$$= 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Properties of Arithmetic, Geometric, Harmonic means between two given numbers

Let *A*, *G* and *H* be arithmetic, geometric and harmonic means of two numbers *a* and *b*.

Then, $A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$.

These three means possess the following properties :

(1) $A \geq G \geq H$

$A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

$\therefore\ A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2} \geq 0 \Rightarrow A \geq G$ …..(i)

$G - H = \sqrt{ab} - \frac{2ab}{a+b} = \sqrt{ab}\left(\frac{a+b-2\sqrt{ab}}{a+b}\right) = \frac{\sqrt{ab}}{a+b}(\sqrt{a} - \sqrt{b})^2 \geq 0$

$\Rightarrow G \geq H$ …..(ii)

From (i) and (ii), we get $A \geq G \geq H$.

Note that the equality holds only when *a* = *b*.

(2) *A*, *G*, *H* from a G.P., *i.e.*, $G^2 = AH$

$AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = (\sqrt{ab})^2 = G^2$. Hence, $G^2 = AH$

(3) The equation having *a* and *b* as its roots is

$$x^2 - 2Ax + G^2 = 0$$

The equation having *a* and *b* its roots is

$$x^2 - (a+b)x + ab = 0$$

$\Rightarrow x^2 - 2Ax + G^2 = 0$, $\left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$.

The roots a, b are given by $A \pm \sqrt{A^2 - G^2}$.

(4) If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

where $A = \frac{a+b+c}{3}, G = (abc)^{1/3}$ and $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$

$\Rightarrow$ $a + b + c = 3A, abc = G^3$ and $\frac{3G^3}{H} = ab + bc + ca$

The equation having a, b, c as its roots is

$x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc = 0$

$\Rightarrow$ $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$.

Relation between A.P., G.P. and H.P.

(1) If A, G, H be A.M., G.M., H.M. between a and b, then

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \begin{cases} A \text{ when } n = 0 \\ G \text{ when } n = -1/2 \\ H \text{ when } n = -1 \end{cases}$$

(2) If A_1, A_2 be two A.M.'s; G_1, G_2 be two G.M.'s and H_1, H_2 be two H.M.'s between two numbers a and b, then

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$$

(3) **Recognization of A.P., G.P., H.P. :** If a, b, c are three successive terms of a sequence.

If $\frac{a-b}{b-c} = \frac{a}{a}$, then a, b, c are in A.P.

If, $\frac{a-b}{b-c} = \frac{a}{b}$, then a, b, c are in G.P.

If, $\frac{a-b}{b-c} = \frac{a}{c}$, then a, b, c are in H.P.

(4) If number of terms of any A.P./G.P./H.P. is odd, then A.M./G.M./H.M. of first and last terms is middle term of series.

(5) If number of terms of any A.P./G.P./H.P. is even, then A.M./G.M./H.M. of middle two terms is A.M./G.M./H.M. of first and last terms respectively.

(6) If p^{th}, q^{th} and r^{th} terms of a G.P. are in G.P. Then p, q, r are in A.P.

(7) If a, b, c are in A.P. as well as in G.P. then $a = b = c$.

(8) If a, b, c are in A.P., then x^a, x^b, x^c will be in G.P. $(x \neq \pm 1)$.