2.2 GRAPH OF POLYNOMIALS

In algebraic or in set theoretic language the graph of a polynomial f(x) is the collection (or set) of all points **(x, y),** where **y = f(x).** In geometrical or in graphical language the graph of a polynomial f(x) is a smooth free hand curve passing through points $x_1, y_1), (x_2, y_2), (x_3, y_3)$, **..... etc.** where y_1, y_2, y_3,.... are the values of the polynomial **f(x)** at x_1, x_2, x_3,.... respectively.

In order to draw the graph of a polynomial **f(x),** follow the following algorithm.

ALGORITHM:

Step (i) Find the values y_1, y_2, ... y_n of polynomial f(x) on different points x_1, x_2, ... x_n………… and prepare a table that gives values of **y** or **f(x)** for various values of **x.**

x:	x_1	x_2	……	x_n	x_{n+1}	….
y = f(x)	$y_1=f(x_1)$	$y_2=f(x_2)$	….	$Y_n=f(x_n)$	$y_{n+1} = f(x_{n+1})$	…..

Step (ii) Plot that points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$,.....$(x_n, y_n)$.... on rectangular co-ordinate system. In plotting these points use different scales on the X and Y axes.

Step (iii) Draw a free hand smooth curve passing through points plotted in **step 2** to get the graph of the polynomial **f(x).**

(a) Graph of a Linear Polynomial :

Consider a linear polynomial **f(x) = ax + b,** $a \neq 0$ Graph of **y = ax + b** is a straight line. That in why f(x) = ax + b) is called a linear polynomial. Since two points determine a straight line, so only two points need to plotted to draw the line **y = ax + b.** The line represented by **y = ax + b** crosses the X-axis at exactly one point, namely $\left(-\frac{b}{a}, 0\right)$.

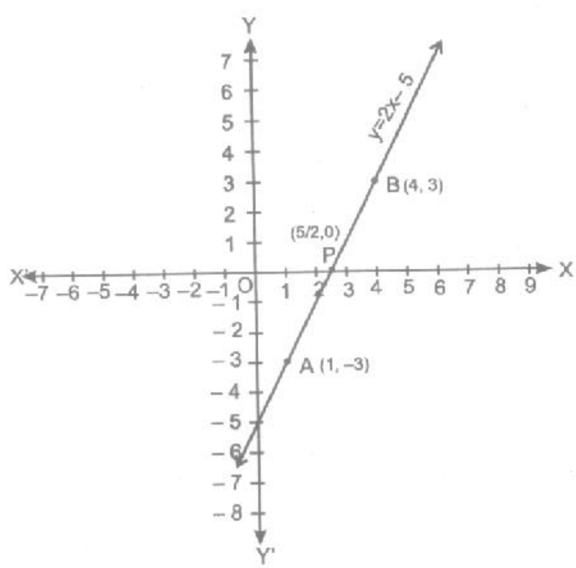
Ex.1 Draw the graph of the polynomial f(x) = 2x - 5. Also, find the coordinates of the point where it crosses X-axis.

Sol. Let y = 2x - 5.

The following table list the values of **y** corresponding to different values of **x.**

x	1	4
y	-3	3

The points A (1, - 3) and B (4, 3) are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graphs of the given polynomial.



(b) Graph of a Quadratic Polynomial :

Let a,b,c be real numbers and $a \neq 0$. Then $f(x) = ax^2 + bx + c$ is known as a quadratic polynomial in x. Graph of the quadratic polynomial i.e. he curve whose equation is $y = ax^2 + bx + c,\ a \neq 0$ Graph of a quadratic polynomial is always a parabola.

Let $y = ax^2 + bx + c$, where $a \neq 0$

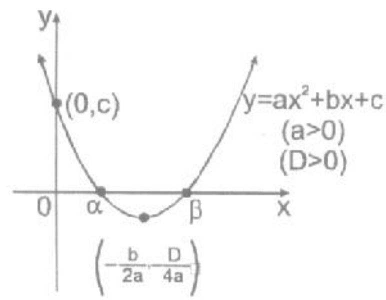
$\Rightarrow \quad 4ay = 4a^2x^2 + 4abx + 4ac$

$\Rightarrow \quad 4ay = 4a^2x^2 + 4abx + b^2 - b^2 + 4ac$

$\Rightarrow \quad 4ay = (2ax + b)^2 - (b^2 - 4ac)$

$\Rightarrow \quad 4ay + (b^2 - 4ac) = (2ax + b)^2 \Rightarrow 4ay + (b^2 - 4ac) = 4a^2(x + b/2a)^2$

$$\Rightarrow \quad 4a\left\{y + \frac{b^2 - 4ac}{4a}\right\} = 4a^2\left(x + \frac{b}{2a}\right)^2$$



$$\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a\left(a + \frac{b}{2a}\right)^2 \qquad \ldots(i)$$

where $D = b^2 - 4ac$ is the discriminate of the quadratic equation.

REMARKS:

Shifting the origin at $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, we have X = x - $\left(-\frac{b}{2a}\right)$ and Y = y - $\frac{(-D)}{4a}$
Substituting these values in (i), we obtain
Y = aX2 ....(ii)
which is the standard equation of parabola

Clearly, this is the equation of a parabola having its vertex at $\left(-\frac{b}{2a}, \frac{D}{4a}\right)$.
The parabola opens upwards or downwards according as a > 0 or a < 0.

SIGN OF QUADRTIC EXPRESSIONS:

Let α be a real root of $\mathbf{ax^2 + bx + c = 0}$. Then $a\alpha^2 + b\alpha + c = 0$ Point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$. Thus, every real root of $ax^2 + bx + c = 0$ represents a point of intersecting of the parabola with the X-axis.

Conversely, if the parabola $y = ax^2 + bx + c$ intersects the X-axis at a point $(\alpha, 0)$ then $(\alpha, 0)$ satisfies the equation $y = ax^2 + bx + c$

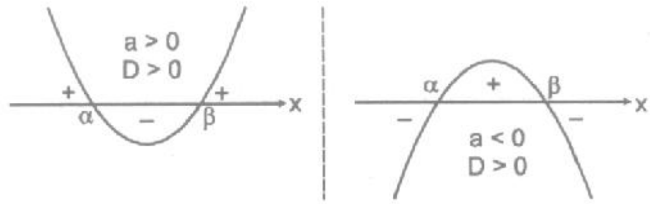
$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0 \qquad [\alpha$ is a real root of $ax^2 + bx + c = 0]$

Thus, the intersection of the parabola $y = ax^2 + bx + c$ with X-axis gives all the real roots of $ax^2 + bx + c = 0$.

Following conclusions may be drawn :-

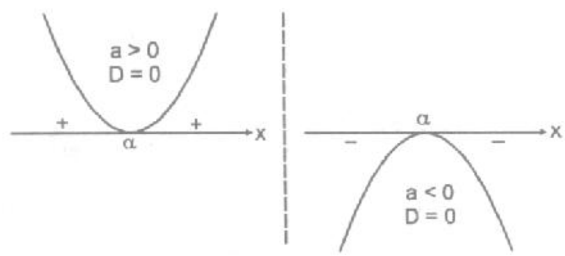
(i) If D>0, the parabola will intersect the x-axis in two distinct points and vice-versa.

The parabola meets x-axis at $\alpha = \frac{-b-\sqrt{D}}{2a}$ and $= \frac{-b+\sqrt{D}}{2a}$.



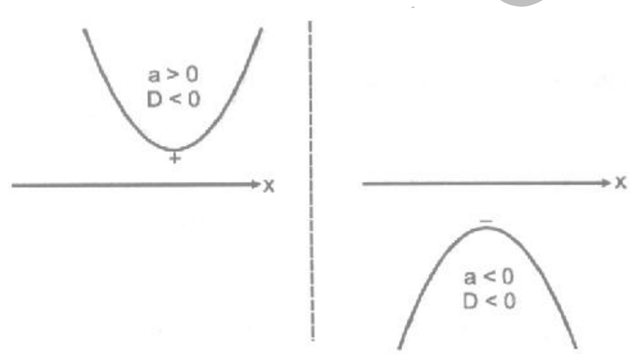
Roots are real & distinct

(ii) If D = 0, the parabola will just touch the x-axis at one point and vice-versa.



Roots are equal

(iii) If D<0, the parabola will not intersect x-axis at all and vice-versa.



Roots are imaginary

REMARKS

$\forall x \in R, y > 0$ **only if a > 0 & D ≡ b2 – 4ac < 0**

$\forall x \in R, y < 0$ **only if a < 0 & D ≡ b2 – 4ac < 0**

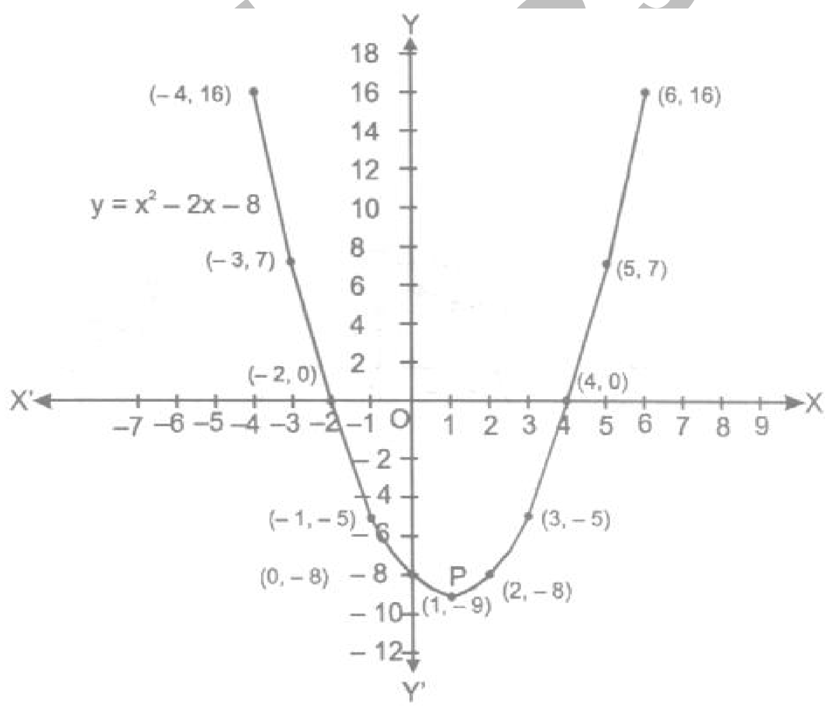
Ex.2 Draw the graph of the polynomial $f(x) = x^2 - 2x - 8$

Sol. Let $y = x^2 - 2x - 8$.

The following table gives the values of y or f(x) for various values of x.

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	-5	-8	-9	-8	-5	0	7	16

Let us plot the points (-4, 16), (-3, 7), (-2, 0), (-1, -5), (0, - 8), (1, - 9), (2, - 8), (3, - 5), (4, 0), (5, 7) and (6, 16) on a graphs paper and draw a smooth free hand curve passing through these points. The curve thus obtained represents the graphs of the polynomial $f(x) = x^2 - 2x - 8$. This is called a parabola. The lowest point P, called a minimum points, is the vertex of the parabola. Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.



Observations :

Fro the graphs of the polynomial $f(x) = x^2 - 2x - 8$, following observations can be drawn :

(i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.

(ii) $D = b^2 - 4ac = 4 + 32 = 36 > 0$. So, the parabola cuts X-axis at two distinct points.

(iii) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get a = 1, b = - 2 and c = - 8.

The vertex of the parabola has coordinates (1, -9) i.e. $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where $D \equiv b^2 - 4ac$.

(iv) The polynomial $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$ is factorizable into two distinct linear factors (x - 4) and (x + 2). So, the parabola cuts X-axis at two distinct points (4, 0) and (-2, 0). the x-coordinates of these points are zeros of f(x).

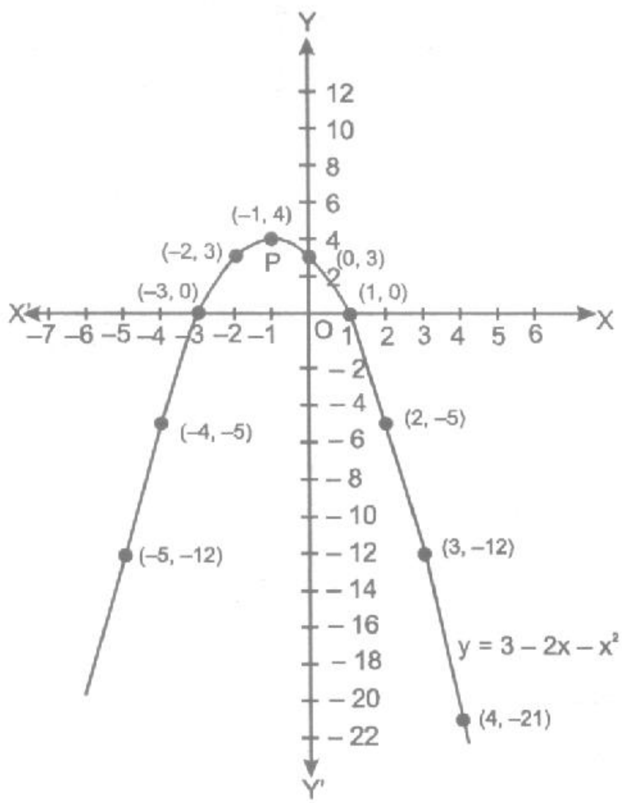
Ex.3 Draw the graphs of the quadratic polynomial $f(x) = 3 - 2x - x^2$.

Sol. Let $y = f(x)$ or, $y = 3 - 2x - x^2$.

Let us list a few values of $y = 3 - 2x - x^2$ corresponding to a few values of x as follows :

x	-5	-4	-3	-2	-1	0	1	2	3	4
$y = 3 - 2x - x^2$	-12	-5	0	3	4	3	0	-5	-12	-21

Thus, the following points lie on the graph of the polynomial $y = 2 - 2x - x^2$:
(-5, -12), (-4, -5), (-3, 0), (-2, 4), (-1, 4), (0, 3), (1, 0), (2, - 5), (3, -12) and (4, - 21).
Let plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graphs of $y = 3 - 2x - x^2$. The curve thus obtained represents a parabola, as shown in figure. The highest point P(-1, 4), is called a maximum points, is the vertex of the parabola. Vertical line through P is the axis of the parabola. Clearly, parabola is symmetric about the axis.



Observations :

Following observations from the graph of the polynomial $f(x) = 3 - 2x - x^2$ is as follows :

(i) The coefficient of x^2 in $f(x) = 3 - 2x - x^2$ is - 1 i.e. a negative real number and so the parabola opens downwards.

(ii) $D \equiv b^2 - 4ax = 4 + 12 = 16 > 0$. So, the parabola cuts x-axis two distinct points.

(iii) On comparing the polynomial 3 - 2x - x^2 with ax^2 + bc + c, we have a = - 1, b = - 2 and c = 3. The vertex of the parabola is at the point (-1, 4) i.e. at $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where D = b^2 - 4ac.

(iv) The polynomial f(x) = 3 - 2x - x^2 = (1 - x) (x + 3) is factorizable into two distinct linear factors (1 - x) and (x + 3). So, the parabola cuts X-axis at two distinct points (1, 0) and (-3, 0). The co-ordinates of these points are zeros of f(x).

(c) GRAPH OF A CUBIC POLYNOMIAL:

Graph of a cubic polynomial does not have a fixed standard shape. Cubic polynomial graphs will always cross X-axis at least once and at most thrice.

Ex.4 Draw the graphs of the polynomial f(x) = x^3 - 4x.

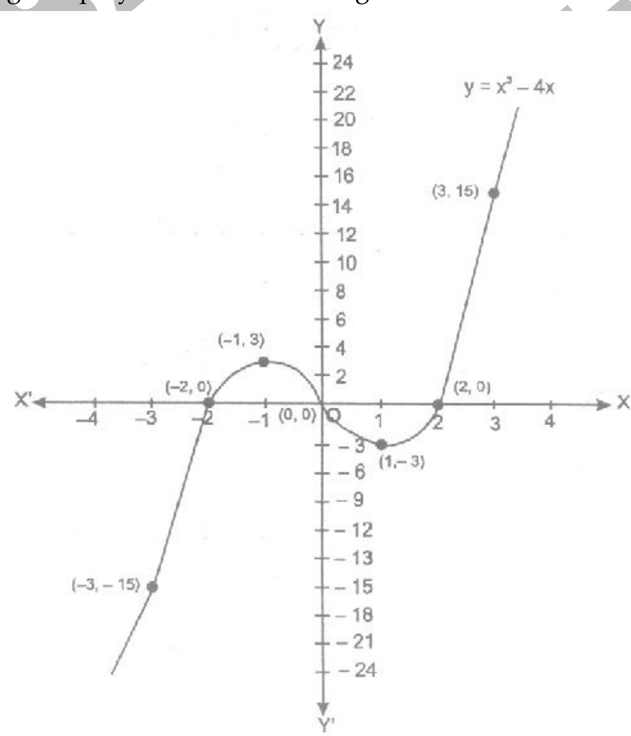
Sol. Let y = f(x) or, y = x^2 - 4x.

The values of y for variable value of x are listed in the following table :

x	-3	-2	-1	0	1	2	3
$y = x^3 - 4x$	-15	0	3	0	-3	0	15

Thus, the curve y = x^3 - 4x passes through the points (-3, -15), (-2, 0), (-1, 3), (0 ,0), (1, -3), (2, 0), (3, 15), (4, 48) etc.

Plotting these points on a graph paper and drawing a free hand smooth curve through these points, we obtain the graph of the given polynomial as shown figure.



Observations :

For the graphs of the polynomial f(x) = x^3 - 4x, following observations are as follows :-

(i) The polynomial $f(x) = x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$ is factorizable into three distinct linear factors. The curve $y = f(x)$ also cuts X-axis at three distinct points.

(ii) We have, $f(x) = x(x - 2)(x + 2)$

Therefore 0, 2 and -2 are three zeros of $f(x)$. The curve $y = f(x)$ cuts X-axis at three points O (0, 0), P(2, 0) and Q (-2, 0).