2.4 RELATIONSHIP BETWEEN ZEROS AND COEFFICIENTS OF A CUBIC POLYNOMIAL

Let α, β, γ be the zeros of a cubic polynomial f(x) = $ax^3 + bx^2 + cx + d$, $a \neq 0$ Then, by factor theorem, $a - \alpha, x - \beta$ and $x - \gamma$ are factors of f(x). Also, f(x) being a cubic polynomial cannot have more than three linear factors.

$\therefore \quad f(x) = k(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = k\ (x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma\}$

$\Rightarrow \quad ax^3 + bx^2 + cx + d = k\,x^3 - k\ (\alpha+\beta+\gamma)x^2 + k(\alpha\beta+\beta\gamma+\gamma\alpha)x - k\alpha\beta\gamma$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$a = k,\ b = -k\ (\alpha+\beta+\gamma), c = (\alpha\beta+\beta\gamma+\gamma\alpha) \text{ and } d = -k(\alpha\beta\gamma)$$

$\Rightarrow \quad \alpha+\beta+\gamma = -\dfrac{b}{a}$

$\Rightarrow \quad \alpha\beta+\beta\gamma+\gamma\alpha = \dfrac{c}{a}$

And, $\quad \alpha\beta\gamma = -\dfrac{d}{a}$

$\Rightarrow$ Sum of the zeros $= -\dfrac{b}{a} = -\dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$\Rightarrow$ Sum of the products of the zeros taken two at a time $= \dfrac{c}{a} = \dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

$\Rightarrow$ Product of the zeros $= -\dfrac{d}{a} = -\dfrac{\text{Constant term}}{\text{Coefficient of } x^3}$

REMARKS:

Cubic polynomial having α, β and γ as its zeros is given by

$f(x) = k\ (x-\alpha)(x-\beta)(x-\gamma)$

$f(x) = k\ \{x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma\}$ where k is any non-zero real number.

Ex.8 Verify that $\frac{1}{2}, 1 - 2$ are zeros of cubic polynomial $2x^3 + x^2 - 5x + 2$. Also verify the relationship between, the zeros and their coefficients.

Sol. $f(x) = 2x^3 + x^2 - 5x + 2$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2$$

$f(1) = 2()^3 + (1)^2\ 5(1) + 2 = 2 + 1 - 5 + 2 = 0.$

$f(-2) = 2(-2)^3 + (-2)^2 - 5(-2) + 2 = -16 + 4 + 10 + 2 = 0.$

Let $\alpha = \frac{1}{2}, \beta = 1$ and $\gamma = -2$

Now, Sum of zeros $= \alpha + \beta + \gamma$

$= \frac{1}{2} + 1 - 2$

$= -\frac{1}{2}$

Also, sum of zeros $= -\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}$

$= -\frac{1}{2}$

So, sum of zeros $= \alpha + \beta + \gamma = -\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}$

Sum of product of zeros taken two at a time $= \alpha\beta + \beta\gamma + \gamma\alpha$

$= \frac{1}{2} \times 1 + 1 \times (-2) + (-2) \times \frac{1}{2}$

$= -\frac{5}{2}$

Also, $\beta\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = \frac{-5}{2}$

So, sum of product of zeros taken two at a time = $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

Now, Product of zeros = $\alpha\beta\gamma$

$= \left(\frac{1}{2}\right)(1)(-2) = -1$

Also, product of zeros = $\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

$= \frac{-2}{2} = -1$

$\therefore$ Product zeros = $\alpha\beta\gamma = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

Ex.9 Find a polynomial with the sum, sum of the product of its zeros taken two at a time, and product its zeros as 3, -1 and -3 respectively.

Sol. Given $\alpha + \beta + \gamma = 3, \alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = -3$

So, polynomial $f(x) = k\ \{x^3 - x^2\ (\alpha + \beta + \gamma) + x(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma\ \}$

$f(x) = k\ \{x^3 - 3x^2 - x + 3\}$, where k is any non-zero real number.