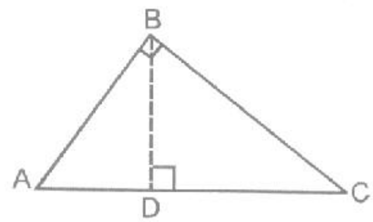
6.4 PYTHAGOREOUS THEOREM

Statement : In a right triangle, the square of the hypotenuse is equal to the sum of the square of the other two sides.

Given : A right triangle ABC, right angled at B.



To prove : $AC^2 = AB^2 + BC^2$

Construction : $BD \perp AC$

Proof : Δ ADB & ΔABC

$\angle DAB = \angle CAB$ [Common]

$\angle BDA = \angle CBA$ [90^0 each]

So, $\Delta ADB \sim \Delta ABC$ [By AA similarity]

$\frac{AD}{AB} = \frac{AB}{AC}$ [Sides are proportional]

or, $AD \cdot AC = AB^2$ .....(i)

Similarly $\Delta BDC \sim \Delta ABC$

So, $\frac{CD}{BC} = \frac{BC}{AC}$

or $CD \cdot AC = BC^2$ .....(ii)

Adding (i) and (ii),

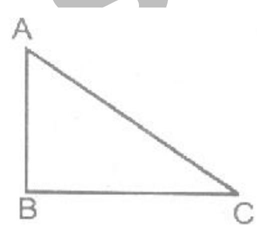
$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

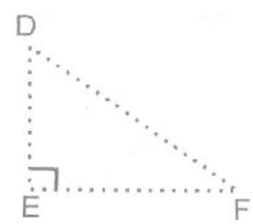
or, $AC (AD + CD) = AB^2 + BC^2$

or $AC.AC = AB^2 + BC^2$

or, $\mathbf{AC^2 = AB^2 + BC^2}$ **Hence Proved.**

(a) Converse of Pythagoreans Theorem:





Statement: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Given: A triangle ABC such that $AC^2 = AB^2 + BC^2$

Construction: Construct a triangle DEF such that DE = AB, EF = BC and $\angle E = 90^0$

Proof: In order to prove that $\angle B = 90.^0$, it is sufficient to show $\Delta ABC \sim \Delta DEF$. For this we proceed as follows Since Δ DEF is a right - angled triangle with right angle at E. Therefore, by Pythagoras theorem, we have

$DF^2 = DE^2 + EF^2$

$\Rightarrow$ $DF^2 = AB^2 + BC^2$ [$\therefore$ DE = AB and EF = BC (By construction)]

$\Rightarrow$ $DF^23 = AC^2$ [$\therefore AB^2 + BC^2 = AC^2$ (Given)]

$\Rightarrow$ DF = AC .....(i)

Thus, in Δ ABC and Δ DEF, we have

AB = DE, BC = EF [By construction]

And AC = DF [From equation (i)]

∴ ΔABC ≅ Δ DEF [By SSS criteria of congruency]

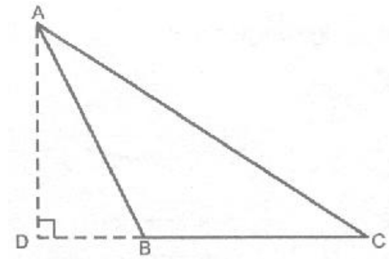
⇒ $\angle B = \angle E = 90^0$

Hence, ΔABC is a right triangle, right angled at B.

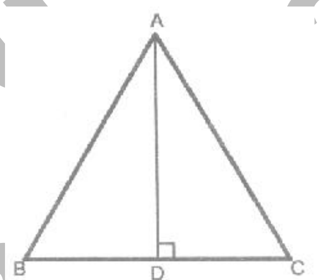
(b) Some Results Deduced From Pythagoreans Theorem :

(i) In the given figure aABC is an obtuse triangle, obtuse angled at **B.** If **AD ⊥ CD,**

then $\mathbf{AC^2 = AB^2 + BC^2 + 2BC\ .\ BC}$



(ii) In the given figure, if ĎB of aABC is an acute angle and **AD ⊥ BC,** then $\mathbf{AC^2 = AB^2 + BC^2 - 2BC\ .\ BD}$



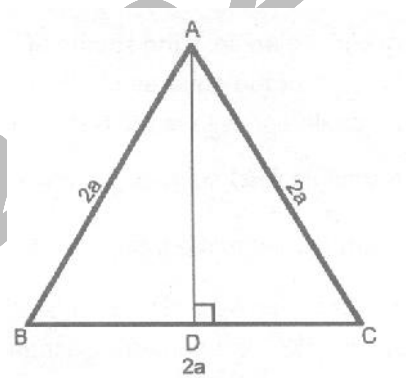
(iii) In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

(iv) Three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares o the medians of the triangle.

Ex.9 In a ΔABC, AB = BC = CA = 2a and AD ⊥ BC. Prove that **[CBSE - 2002]**

(i) $AD = a\sqrt{3}$ **(ii)** area (ΔABC) = $\sqrt{3}\,a^2$

Sol. (i) Here, AD ⊥ BC.



Clearly, ΔABC is an equilateral triangle.

Thus, in ΔABD and ΔACD

AD = AD [Common]

∠ADB = ∠ADC [90^0 each]

And AB = AC

∴ by RHS congruency condition

ΔABD ≅ ΔACD

⇒ BD = DC = a

Now, ΔABD is a right angled triangle

∴ $AD = \sqrt{AB^2 - BD^2}$ [Using Pythagoreans Theorem]

$AD = \sqrt{4a^2 - a^2} = \sqrt{3}\,a$ or $a\sqrt{3}$

(ii) Area (ΔABC) = $\frac{1}{2}$ × BC × AD

$$= \frac{1}{2} \times 2a \times a\sqrt{3}$$

$$= a^2\sqrt{3}$$

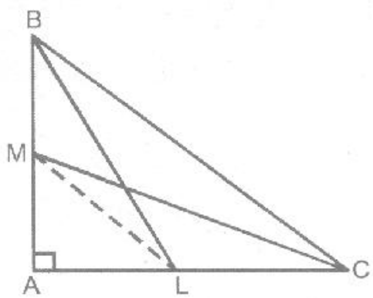
Ex.10 BL and Cm are medians of ΔABC right angled at A. Prove that $4(BL^2 + CM^2) = 5\,BC^2$ **[CBSE-2006]**

Sol. In ΔBAL

$BL^2 = AL^2 + AB^2$ ....(i) [Using Pythagoreans theorem]

and In ΔCAM

$CM^2 = AM^2 + AC^2$ .....(ii) [Using Pythagoreans theorem]



Adding (1) and (2) and then multiplying by 4, we get

$4(BL^2 + CM^2) = 4(AL^2 + AB^2 + AM^2 + AC^2)$

$= 4\{AL^2 + AM^2 + (AB^2 + AC^2)\}$ [$\therefore$ Δ ABC is a right triangle]

$= 4(AL^2 + AM^2 + BC^2)$

$= 4(ML^2 + BC^2)$ [$\therefore$ ΔLAM is a right triangle]

$= 4ML^2 + 4\,BC^2$

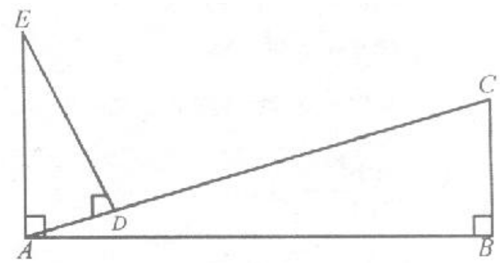
[A line joining mid-points of two sides is parallel to third side and is equal to half of it, ML = BC/2]

$= BC^2 + 4BC^2 = 5BC^2$ **Hence proved.**

Ex.11 In the given figure, BC $\perp$ AB, AE $\perp$ AB and DE $\perp$ AC. Prove that DE.BC = AD.AB.

Sol. In ΔABC and ΔEDA,

We have



$\angle$ABC = ΔADE [Each equal to 90^0]

$\angle$ACB = $\angle$EAD [Alternate angles]

$\therefore$ By AA Similarity

ΔABC ~ ΔEDA

$\Rightarrow \frac{BC}{AB} = \frac{AD}{DE}$

$\Rightarrow$ DE.BC = AD.AB. **Hence Proved.**

Ex.12 O is any point inside a rectangle ABCD (shown in the figure). Prove that $OB^2 + OD^2 = OA^2 + OC^2$

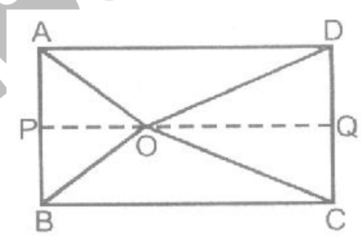
Sol. Through O, draw PQ||BC so that P lies on A and Q lies on DC. **[CBSE - 2006]**

Now, PQ||BC

Therefore, PQ $\perp$ AB and PQ $\perp$ DC [$\angle$B = 90^0 and $\angle$ C = 90^0]

So, $\angle$ BPQ = 90^0 and $\angle$ CQP = 90^0

Therefore, BPQC and APQD are both rectangles.



Now, from Δ OPB,

$OB^2 = BP^2 + OP^2$ ....(i)

Similarly, from Δ ODQ,

$OD^2 = OQ^2 + DQ^2$ ....(ii)

From Δ OQC, we have

$OC^2 = OQ^2 + CQ^2$ ...(iii)

And form Δ OAP, we have

$OA^2 = AP^2 + OP^2$ ....(iv)

Adding (i) and (ii)

$OB^2 + OD^2 = BP^2 + OP^2 + OQ^2 + DQ^2$

$= CQ^2 + OP^2 + OQ^2 + AP^2$

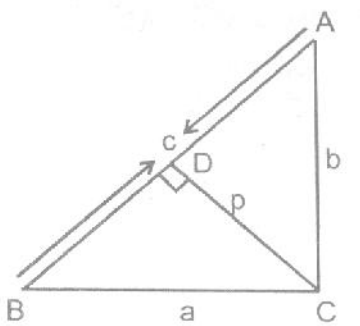
[As BP = CQ and DQ = AP]

$= CQ^2 + OQ^2 + OP^2 + AP^2$

$= OC^2 + OA^2$ [From (iii) and (iv)] **Hence Proved.**

Ex.13 ABC is a right triangle, right-angled at C. Let BC = a, CA b, AB = c and let p be the length of perpendicular form C on AB, prove that

(i) cp = ab (ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$



Sol. Let CD ⊥ AB. Then CD = p

$\therefore$ Area of $\Delta ABC = \frac{1}{2}$ (Base × height)

$= \frac{1}{2}(AB \times CD) = \frac{1}{2}cp$

Also,

Area of Δ ABC $= \frac{1}{2}(BC \times AC) = \frac{1}{2}ab$

$\therefore \frac{1}{2}cp = \frac{1}{2}ab$

$\Rightarrow$ CP = AB.

(ii) Since Δ ABC is a right triangle, right angled at C.

$\therefore AB^2 = BC^2 + AC^2$

$\Rightarrow c^2 = a.. + b^2$

$\Rightarrow \left(\frac{ab}{p}\right)^2 = a^2 + b^2$ $\left[\because cp = ab \Rightarrow c = \frac{ab}{p}\right]$

$\Rightarrow \frac{a^2b^2}{p^2} = a^2 + b^2$

$\Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$

$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Ex.14 In an equilateral triangle ABC, the side B is trisected at D. Prove that 9 $AD^2 = 7AB^2$.

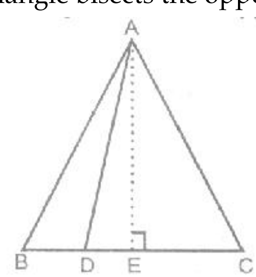
Sol. ABC be can equilateral triangle and D be point on BC such that **[CBSE - 2005]**

$BC = \frac{1}{3}BC$ (Given)

Draw AE ⊥ BC, Join AD.

BE = EC (Altitude drown from any vertex of an equilateral triangle bisects the opposite side)

So, $BE = EC = \frac{BC}{2}$



In Δ ABC

$AB^2 = AE^2 + EB^2$ .....(i)

$AD^2 = AE^2 + ED^2$ ....(ii)

From (i) and (ii)

$AB^2 = AD^2 - ED^2 + EB^2$

$AB^2 = AD^2 - \frac{BC^2}{36} + \frac{BC^2}{4}$ $\left(\therefore BD + DE = \frac{BC}{2} \Rightarrow \frac{BC}{3} + DE = \frac{BC}{2} \Rightarrow DE = \frac{BC}{6}\right)$

$$AB^2 + \frac{BC^2}{36} - \frac{BC^2}{4} = AD^2 \qquad (\because EB = \frac{BC}{2})$$

$$AB^2 + \frac{AB^2}{36} - \frac{AB^2}{4} = AD^2 \qquad (\because AB = BC)$$

$$\frac{36AB^2 + AB^2 - 9AB^2}{36} = AD^2 \quad \Rightarrow \quad \frac{28AB^2}{36} = AD^2$$

$$7AB^2 = 9AD^2$$