CHAPTER – 1
Relations

Definition

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as $a\,R\,b$. If $(a, b) \in R$, we write it as $a\,R\,b$.

(1) **Total number of relations:** Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subset of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ¤ and the universal relation $A \times B$ are trivial relations from A to B.

(2) **Domain and range of a relation:** Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

Inverse relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Clearly $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$. Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Example: Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$.

Then, (i) $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$

(ii) Dom $(R) = \{a, b, c\}$ = Range (R^{-1})

(iii) Range $(R) = \{1, 3\}$ = Dom (R^{-1})

Types of relations

(1) **Reflexive relation:** A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1); (1, 3)\}$

Then R is not reflexive since $3 \in A$ but $(3, 3) \notin R$

A reflexive relation on A is not necessarily the identity relation on A.

The universal relation on a non-void set A is reflexive.

(2) **Symmetric relation:** A relation R on a set A is said to be a symmetric relation *iff* $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e., $aRb \Rightarrow bRa$ for all $a, b \in A$.

it should be noted that R is symmetric iff $R^{-1} = R$

The identity and the universal relations on a non-void set are symmetric relations.

A reflexive relation on a set A is not necessarily symmetric.

(3) **Anti-symmetric relation:** Let A be any set. A relation R on set A is said to be an anti-symmetric relation *iff* $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Thus, if $a \neq b$ then a may be related to b or b may be related to a, but never both.

(4) **Transitive relation:** Let A be any set. A relation R on set A is said to be a transitive relation *iff* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Transitivity fails only when there exists a, b, c such that $a\,R\,b$, $b\,R\,c$ but $a \not{R}\, c$.

Example **:** Consider the set $A = \{1, 2, 3\}$ and the relations

$R_1 = \{(1, 2), (1, 3)\}$; $R_2 = \{(1, 2)\}$; $R_3 = \{(1, 1)\}$;

$R_4 = \{(1, 2), (2, 1), (1, 1)\}$

Then R_1, R_2, R_3 are transitive while R_4 is not transitive since in $R_4, (2, 1) \in R_4; (1, 2) \in R_4$ but $(2, 2) \notin R_4$.

The identity and the universal relations on a non-void sets are transitive.

(5) **Identity relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only. Every identity relation will be reflexive, symmetric and transitive.

Example **:** On the set = $\{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A .

It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

(6) **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A *iff*

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$

(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$

(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Congruence modulo **(m):** Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b$ (mod m).

Thus $a \equiv b$ (mod m) $\Leftrightarrow a - b$ is divisible by m. For example, $18 \equiv 3$ (mod 5) because 18 – 3 = 15 which is divisible by 5. Similarly, $3 \equiv 13$ (mod 2) because 3 – 13 = –10 which is divisible by 2. But 25 ≠ 2 (mod 4) because 4 is not a divisor of 25 – 3 = 22.

The relation "Congruence modulo m" is an equivalence relation.

Equivalence classes of an equivalence relation

Let R be equivalence relation in $A(\neq \phi)$. Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{\bar{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x\,R\,a\}$.

It is easy to see that

(1) $b \in [a] \Rightarrow a \in [b]$

(2) $b \in [a] \Rightarrow [a] = [b]$

(3) Two equivalence classes are either disjoint or identical.

Composition of relations

Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of R and S.

For example, if $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C=\{p, q, r, s\}$ be three sets such that $R = \{(1, a), (2, b), (1, c), (2, d)\}$ is a relation from A to B and $S = \{(a, s), (b, r), (c, r)\}$ is a relation from B to C. Then SoR is a relation from A to C given by $SoR = \{(1, s) (2, r) (1, r)\}$

In this case RoS does not exist.

In general $RoS \neq SoR$. Also $(SoR)^{-1} = R^{-1}oS^{-1}$.

Functions

If A and B are two non-empty sets, then a rule f which associated to each $x \in A$, a unique number $y \in B$, is called a function from A to B and we write, $f : A \to B$.

Some important definitions

(1) **Real numbers:** Real numbers are those which are either rational or irrational. The set of real numbers is denoted by R.

(2) **Related quantities:** When two quantities are such that the change in one is accompanied by the change in other, i.e., if the value of one quantity depends upon the other, then they are called related quantities.

(3) **Variable:** A variable is a symbol which can assume any value out of a given set of values.

(i) **Independent variable:** A variable which can take any arbitrary value, is called independent variable.

(ii) **Dependent variable:** A variable whose value depends upon the independent variable is called dependent variable.

(4) **Constant:** A constant is a symbol which does not change its value, *i.e.*, retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc. There are two types of constant, absolute constant and arbitrary constant.

(5) **Absolute value:** The absolute value of a number x, denoted by $|x|$, is a number that satisfies the conditions

$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$. We also define $|x|$ as follows,

$|x|$= maximum $\{x, -x\}$ or $|x| = \sqrt{x^2}$.

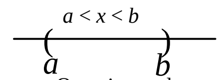
(6) **Fractional part:** We know that $x \geq [x]$. The difference between the number 'x' and it's integral value '$[x]$' is called the fractional part of x and is symbolically denoted as $\{x\}$. Thus, $\{x\} = x - [x]$ *e.g.*, if x = 4.92 then $[x] = 4$ and $\{x\}$= 0.92.

Fractional part of any number is always non-negative and less than one.

Intervals

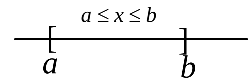
There are four types of interval:

(1) **Open interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by $]a, b[$ or (a, b). Thus, $]a, b[$ or $(a, b)=\{x \in R : a < x < b\}$.



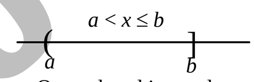
Open interval

(2) **Closed interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by $[a, b]$. Thus, $[a, b] = \{x \in R : a \le x \le b\}$



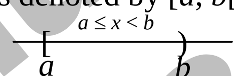
Closed interval

(3) **Open-Closed interval :** It is denoted by $]a, b]$ or $(a, b]$ and $]a, b]$ or $(a, b] = \{x \in R : a < x \le b\}$.



Open closed interval

(4) **Closed-Open interval :** It is denoted by $[a, b[$ or $[a, b)$ and $[a, b[$ or $[a, b) = \{x \in R : a \le x < b\}$



Closed open interval

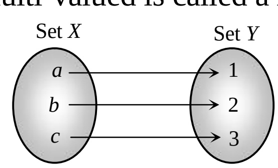
Definition of function

(1) Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f : X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

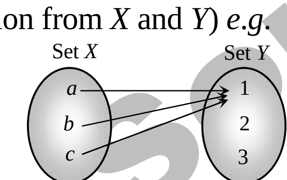
Two things should always be kept in mind:

(i) A mapping $f : X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

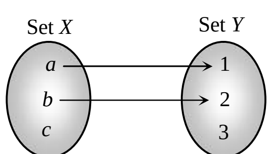
(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from X and Y) *e.g.*



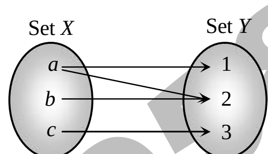
Function



Function

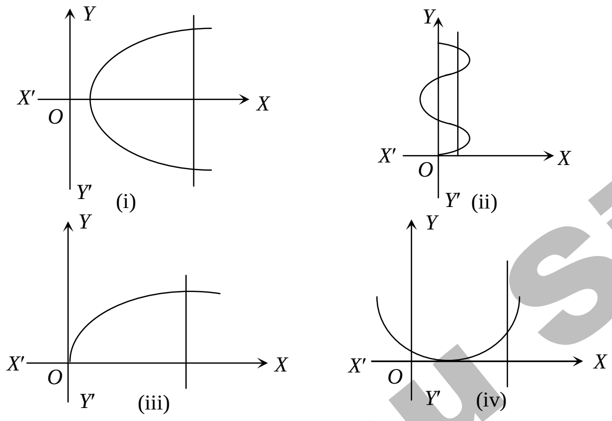


Not function



Not function

(2) Testing for a function by vertical line test : A relation $f : A \to B$ is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



(3) **Number of functions:** Let *X* and *Y* be two finite sets having *m* and *n* elements respectively. Then each element of set *X* can be associated to any one of *n* elements of set *Y*. So, total number of functions from set *X* to set *Y* is n^m .

(4) Value of the function: If $y = f(x)$ is a function then to find its values at some value of x, say $x = a,$ we directly substitute x = a in its given rule $f(x)$ and it is denoted by $f(a)$.

e.g. If $f(x) = x^2 + 1,$ then $f(1) = 1^2 + 1 = 2,\ \ f(2) = 2^2 + 1 = 5,\ \ f(0) = 0^2 + 1 = 1$ *etc*.

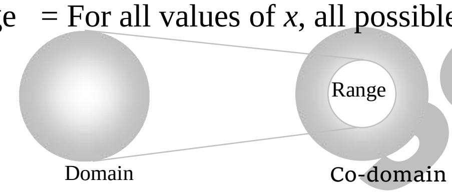
Domain, co-domain and range of function

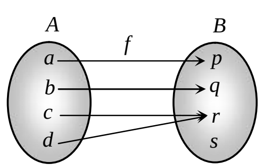
If a function *f* is defined from a set *A* to set *B* then for $f : A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

In other words, we can say

Domain = All possible values of *x* for which *f*(*x*) exists.

Range = For all values of *x*, all possible values of *f*(*x*).





Domain $= \{a, b, c, d\} = A$

Co-domain $= \{p, q, r, s\} = B$

Range $= \{p, q, r\}$

(1) **Methods for finding domain and range of function**

(i) **Domain**

(a) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0. Denominator $\neq 0$.

If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$

(ii) **Range:** Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

(a) If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(b) If domain $\in R$ or R – [some finite points]. Then express x in terms of y. From this find y for x to be defined (*i.e.*, find the values of y for which x exists).

(c) If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

Algebra of functions

(1) **Scalar multiplication of a function:** $(c\,f)(x) = c\,f(x)$, where c is a scalar. The new function $c\,f(x)$ has the domain X_f.

(2) **Addition/subtraction of functions**

$(f \pm g)(x) = f(x) \pm g(x)$. The new function has the domain X.

(3) **Multiplication of functions**

$(fg)(x) = (g\,f)(x) = f(x)g(x)$. The product function has the domain X.

(4) **Division of functions :**

(i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain X, except for the values of x for which $g(x) = 0$.

(ii) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain X, except for the values of x for which $f(x) = 0$.

(5) **Equal functions :** Two function f and g are said to be equal functions, if and only if

(i) Domain of f = Domain of g

(ii) Co-domain of f = Co-domain of g

(iii) $f(x) = g(x)\ \forall x \in$ their common domain

(6) **Real valued function :** If R, be the set of real numbers and A, B are subsets of R, then the function $f : A \to B$ is called a real function or real –valued function.

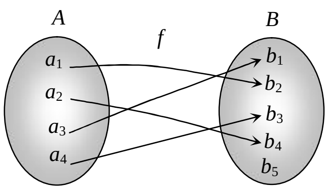
Kinds of function

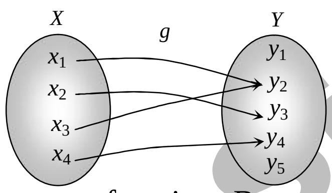
(1) **One-one function (injection) :** A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus, $f : A \to B$ is one-one.

$a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

e.g. Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams.





Clearly, $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function *g*.

(i) **Method to check the injectivity of a function**

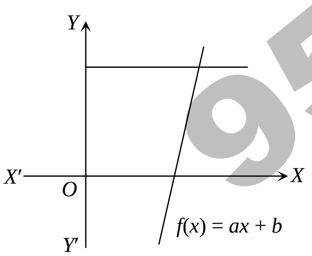
Step I : Take two arbitrary elements *x*, *y* (say) in the domain of *f*.

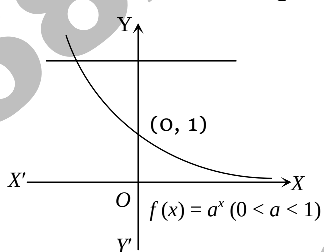
Step II : Put $f(x) = f(y)$.

Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection). Otherwise not.

If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

If the graph of the function $y = f(x)$ is given and each line parallel to *x*-axis cuts the given curve at maximum one point then function is one-one. *e.g.*



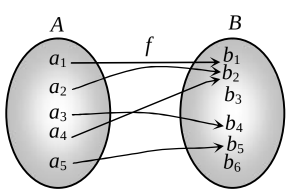


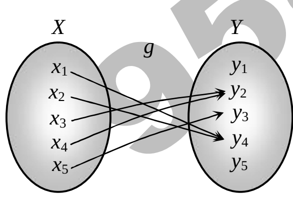
(ii) **Number of one-one functions (injections):** If *A* and *B* are finite sets having *m* and *n* elements respectively, then number of one-one functions from A to B $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(2) **Many-one function:** A function $f : A \to B$ is said to be a many-one function if two or more elements of set *A* have the same image in *B*.

Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

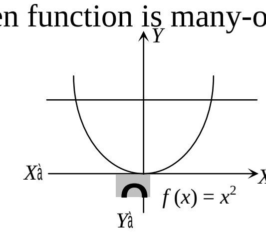
In other words, $f : A \to B$ is a many-one function if it is not a one-one function.

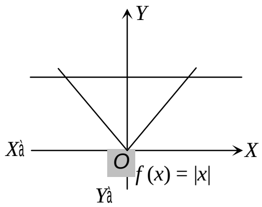




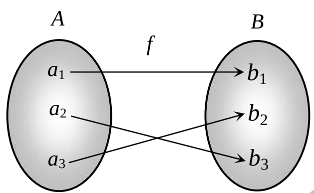
• If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

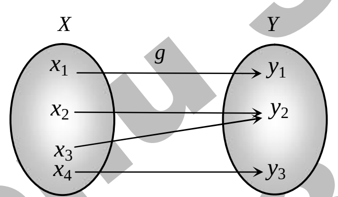
• If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.





(3) **Onto function (surjection):** A function $f: A \to B$ is onto if each element of B has its pre-image in A. Therefore, if $f^{-1}(y) \in A, \forall y \in B$ then function is onto. In other words, Range of f = Co-domain of f. *e.g.* The following arrow-diagram shows onto function.

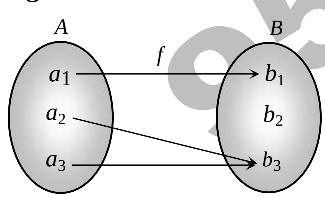


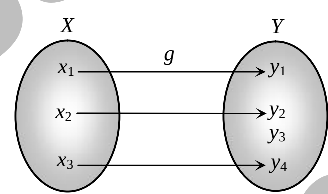


Number of onto function (surjection) : If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r r^m$.

(4) **Into function:** A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f: A \to B$ is an into function if it is not an onto function *e.g.* The following arrow-diagram shows into function.





(i) **Method to find onto or into function**

(a) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say).

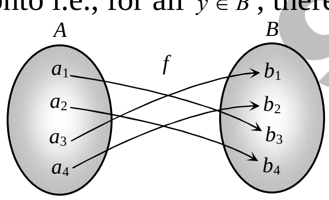
(b) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.

(5) **One-one onto function (bijection):** A function $f: A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f: A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.



Clearly, *f* is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection): If *A* and *B* are finite sets and $f : A \to B$ is a bijection, then *A* and *B* have the same number of elements. If *A* has *n* elements, then the number of bijection from *A* to *B* is the total number of arrangements of *n* items taken all at a time *i.e.* *n*!.

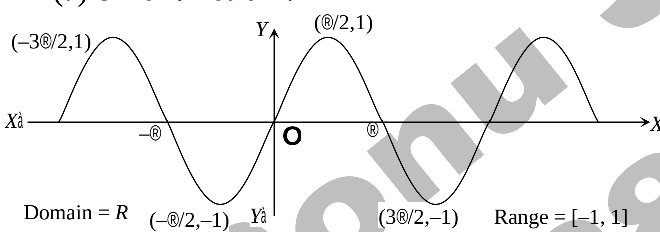
(6) **Algebraic functions:** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, –, × and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}} + 5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4 - 5x + 7$

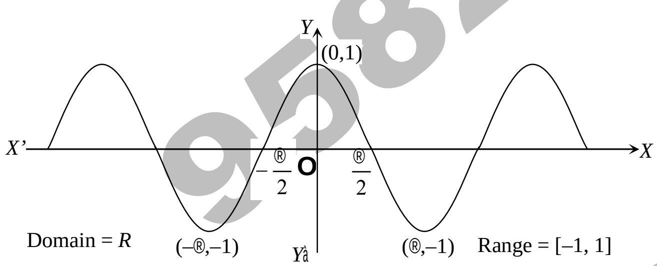
(7) **Transcendental function:** A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric , exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions:** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.

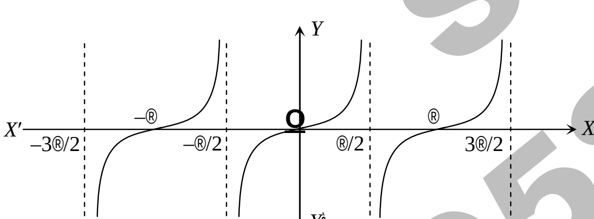
(a) **Sine function :**



(b) **Cosine function :**

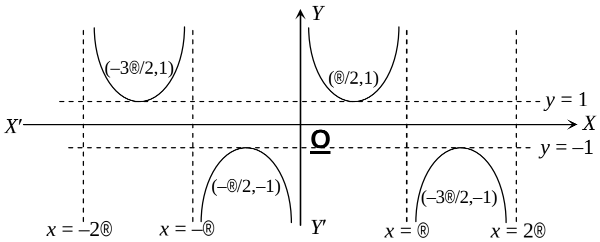


(c) **Tangent function :**



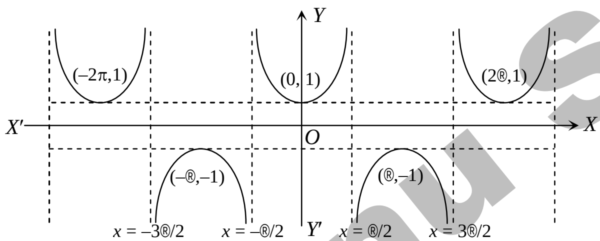
Domain = *R* – {(2*n* +1) ®/2|*n* ∈ *I*}, Range = *R*

(d) **Cosecant function :**



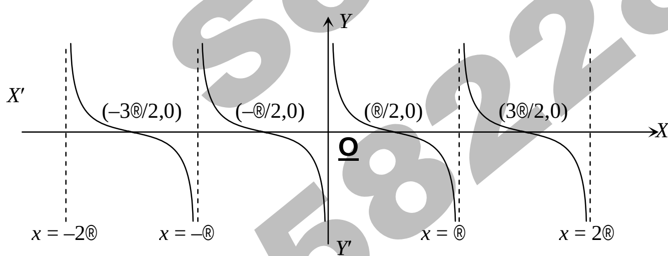
Domain = R – {(n®|n ∈ I}, Range = (– ∞, –1] ∪ [1,∞)

(e) **Secant function :**



Domain = R – {(2n+1)®/2 |n ∈ I}, Range = (– ∞, –1] ∪ [1,∞)

(f) **Cotangent function :**



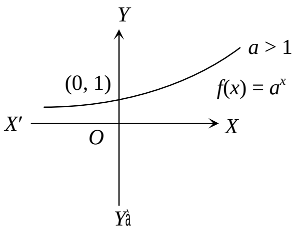
Domain = R – {n®|n ∈ I}, Range = R

(ii) **Inverse trigonometric functions**

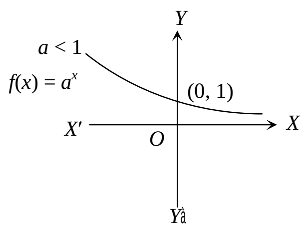
Table : 21.1

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	[–1,1]	[–®/2, ®/2]	$y = \sin^{-1} x$ ⇔ $x = \sin y$
$\cos^{-1} x$	[–1, 1]	[0, ®]	$y = \cos^{-1} x$ ⇔ $x = \cos y$
$\tan^{-1} x$	(–∞,∞) or R	(–®/2, ®/2)	$y = \tan^{-1} x$ ⇔ $x = \tan y$
$\cot^{-1} x$	(–∞,∞) or R	(0, π)	$y = \cot^{-1} x$ ⇔ $x = \cot y$
$\text{cosec}^{-1} x$	R – (–1, 1)	[–®/2, ®/2] – {0}	$y = \text{cosec}^{-1} x$ ⇔ $x = \text{cosec}\ y$
$\sec^{-1} x$	R – (–1, 1)	[0, ®] – [®/2]	$y = \sec^{-1} x$ ⇔ $x = \sec y$

(iii) **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f: R \to (0,\infty)$ defined by $f(x) = a^x$ called exponential function. Its domain is *R* and range is $(0,\infty)$.

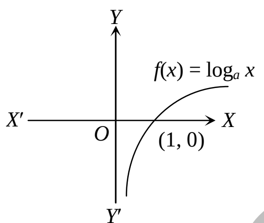


Graph of $f(x) = a^x$, when $a > 1$

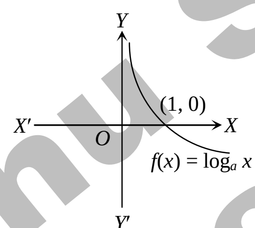


Graph of $f(x) = a^x$, when $a<1$

(iv) **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f:(0,\infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0,\infty)$ and range is *R*.



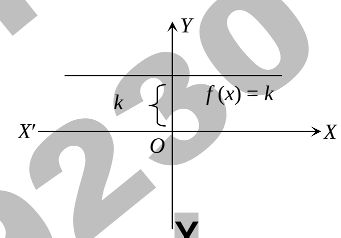
Graph of $f(x) = \log_a x$, when $a > 1$



Graph of $f(x) = \log_a x$, when $a < 1$

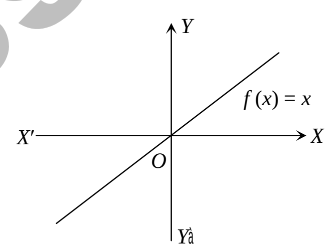
(8) **Explicit and implicit functions :** A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *e.g.* $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a-x)^2 (b-y)^2$ are implicit functions.

(9) **Constant function :** Let *k* be a fixed real number. Then a function *f*(*x*) given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of *f* is the singleton set {*k*}. The graph of a constant function is a straight line parallel to *x*-axis as shown in figure and it is above or below the *x*-axis according as *k* is positive or negative. If *k* = 0, then the straight line coincides with *x*-axis.

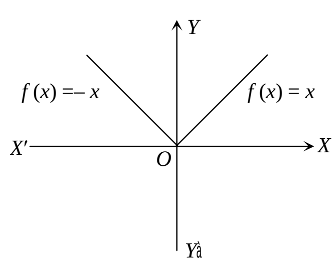


(10) **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on *R*. Clearly, the domain and range of the identity function is *R*.

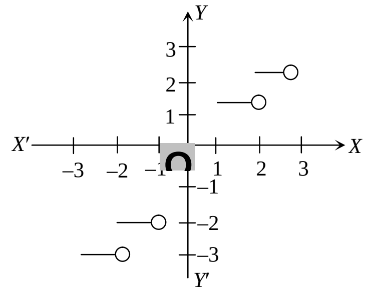
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of *x*-axis.



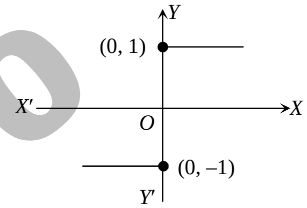
(11) **Modulus function :** The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set *R* of all real numbers and the range is the set of all non-negative real numbers.



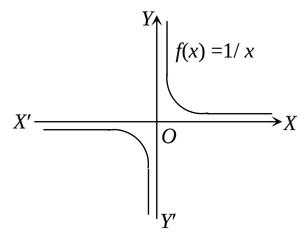
(12) **Greatest integer function:** Let $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x. The domain is R and the range is I. *e.g.* [1.1] = 1, [2.2] = 2, [– 0.9] = –1, [– 2.1] = – 3 etc. The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.



(13) **Signum function :** The function defined by $f(x) = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ or $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is called the signum function. The domain is R and the range is the set {–1, 0, 1}.



(14) **Reciprocal function :** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R - \{0\}$ *i.e.*, the set of all non-zero real numbers. The graph is as shown.



(15) **Power function:** A function $f : R \to R$ defined by, $f(x) = x^{\sim}$, $\sim \in R$ is called a power function.

Table : 21. 2 Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	$\{K\}$
Reciprocal function $\frac{1}{x}$	R_0	R_0
$x^2, \|x\|$	R	$R^+ \cup \{0\}$
$x^3, x\|x\|$	R	R
Signum function	R	{–1, 0, 1}
$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$
$[x]$	R	I
$x - [x]$	R	[0, 1)
$\sqrt{x}$	$[0, \infty)$	$[0, \infty]$
a^x	R	R^+
$\log x$	R^+	R

Function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R-\left\{\pm\frac{®}{2}, \pm\frac{3®}{2},...\right\}$	R
$\cot x$	$R-\{0, \pm ®, \pm 2®,...\}$	R
$\sec x$	$R-\left\{\pm\frac{®}{2}, \pm\frac{3®}{2},...\right\}$	$R-(-1, 1)$
$\text{cosec}\,x$	$R-\{0, \pm ®, \pm 2®,...\}$	$R-(-1, 1)$
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{-®}{2}, \frac{®}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, ®]$
$\tan^{-1} x$	R	$\left(\frac{-®}{2}, \frac{®}{2}\right)$
$\cot^{-1} x$	R	$(0, ®)$
$\sec^{-1} x$	$R-(-1,1)$	$[0,®]-\left\{\frac{®}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1,1)$	$\left[-\frac{®}{2},\frac{®}{2}\right]-\{0\}$

Even and Odd function

(1) **Even function:** If we put $(-x)$ in place of x in the given function and if $f(-x)=f(x)$, $\forall x \in$ domain then function $f(x)$ is called even function. *e.g.* $f(x)=e^x+e^{-x}$, $f(x)=x^2$, $f(x)=x\sin x$, $f(x)=\cos x, f(x)=x^2\cos x$ all are even functions.

(2) **Odd function:** If we put $(-x)$ in place of x in the given function and if $f(-x)=-f(x), \forall x \in$ domain then $f(x)$ is called odd function. *e.g.*, $f(x)=e^x-e^{-x}$, $f(x)=\sin x, f(x)=x^3$, $f(x)=x\cos x$, $f(x)=x^2\sin x$ all are odd functions.

Properties of even and odd function

- The graph of even function is always symmetric with respect to y-axis. The graph of odd function is always symmetric with respect to origin.
- The product of two even functions is an even function.
- The sum and difference of two even functions is an even function.
- The sum and difference of two odd functions is an odd function.
- The product of two odd functions is an even function.
- The product of an even and an odd function is an odd function. It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function. *e.g.* $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.
- The sum of even and odd function is neither even nor odd function.
- Zero function $f(x) = 0$ is the only function which is even and odd both.

Periodic function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x+T)=f(x), \forall x \in$ domain. Here the least positive value of T is called the period of the function.

Composite function

If $f: A \to B$ and $g: B \to C$ are two function then the composite function of f and g, $gof\ A \to C$ will be defined as $gof(x) = g[f(x)], \forall x \in A$

(1) **Properties of composition of function:**

(i) f is even, g is even $\Rightarrow fog$ even function.

(ii) f is odd, g is odd $\Rightarrow fog$ is odd function.

(iii) f is even, g is odd $\Rightarrow fog$ is even function.

(iv) f is odd, g is even $\Rightarrow fog$ is even function.

(v) Composite of functions is not commutative *i.e.*, $fog \neq gof$.

(vi) Composite of functions is associative *i.e.*, $(fog)oh = fo(goh)$

(vii) If $f: A \to B$ is bijection and $g: B \to A$ is inverse of f. Then $fog = I_B$ and $gof = I_A$.

where, I_A and I_B are identity functions on the sets A and B respectively.

(viii) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.

(ix) $fog \neq gof$ but if, $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$.

(x) $gof(x)$ is simply the g-image of $f(x)$, where $f(x)$ is f-image of elements $x \in A$.

(xi) Function gof will exist only when range of f is the subset of domain of g.

(xii) fog does not exist if range of g is not a subset of domain of f.

(xiii) fog and gof may not be always defined.

(xiv) If both f and g are one-one, then fog and gof are also one-one.

(xv) If both f and g are onto, then gof is onto.

Inverse function

If $f: A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1}: B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f: A \to B$.

$f^{-1}: B \to A, f^{-1}(b) = a \Rightarrow f(a) = b$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

For the existence of inverse function, it should be one-one and onto.

Properties of Inverse function:

(1) Inverse of a bijection is also a bijection function.

(2) Inverse of a bijection is unique.

(3) $(f^{-1})^{-1} = f$

(4) If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$.

(5) If $f: A \to B$ is a bijection then $f^{-1}. B \to A$ is an inverse function of f. $f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here I_A, is an identity function on set A, and I_B, is an identity function on set B.